AMITABHA LAHIRI • PALASH B. PAL

A First Book of QUANTUM FIELD THEORY

Second Edition

Alpha Science

Preface to the second edition

There has not been any drastic changes in the present edition. Of course some of the typographical and other small mistakes, posted on the internet page for the book, have been corrected. In addition, we have added occassional comments and clarifications at some places, hoping that they would make the discussions more accessible to a beginner.

The only serious departure from the first edition has been in the notation for spinor solutions of the Dirac equation. In the previous edition, we put a subscript on the spinors which corresponded to their helicities. In this edition, we change the notation towards what is more conventional. This notation has been introduced in §4.3 and explained in detail in §4.6.

We have received many requests for posting the answers to the exercises. In this edition, we have added answers to some exercises which actually require an answer. The web page for the errata of the book is now at

`http://tnp.saha.ernet.in/~pbpal/books/qft/errata.html`

If you find any mistake in the present edition, please inform us through this web page.

It is a pleasure to thank everyone who has contributed to this book by pointing out errors or asking for clarifications. On the errata page we have acknowledged all correspondence which have affected some change. In particular, comments by C. S. Aulakh, Martin Einhorn and Sean Murray were most helpful.

Amitabha Lahiri
Palash B. Pal

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Preface to the first edition

It has been known for more than fifty years that Quantum Field Theory is necessary for describing precision experiments involving electromagnetic interactions. Within the last few decades of the twentieth century it has also become clear that the weak and the strong interactions are well described by interacting quantum fields. Although it is quite possible that at even smaller length scales some other kind of theory may be operative, it is clear today that quantum fields provides the appropriate framework to describe a wide class of phenomena in the energy range covered by all experiments to date.

In this book, we wanted to introduce the subject as a beautiful but essentially simple piece of machinery with a wide range of applications. This book is meant as a textbook for advanced undergraduate or beginning post-graduate students. For this reason, we employ canonical quantization throughout the book. The name of the book is an echo of various children's texts that were popular a long time ago, not a claim to primacy or originality.

Our approach differs from many otherwise excellent textbooks at the introductory level which set up the description of electrons and photons as their goal. For example, decays are rarely discussed, since the electron and the photon are both stable particles. However, decay processes are in some sense simpler than scattering processes, since the former has only one particle in the initial state whereas the latter has two.

We felt that the basic machinery of Feynman diagrams could be introduced through decay processes even before talking about the quantization of spin-1 fields. With that in mind, we start with some introductory material in Chapters 1 and 2 and discuss the quantization of scalar fields in Chapter 3 and of spin-$\frac{1}{2}$ fields in Chapter 4. Unlike many other texts at this level, we use a fermion normalization that should be applicable to massless as well as massive fermions. After that we discuss the generalities of the S-matrix theory in Chapter 5, and the methods of calculating Feynman diagrams, decay rates, scattering cross sections etc. with spin-0 and spin-$\frac{1}{2}$ fields, in Chapters 6 and 7.

The quantization of the spin-1 fields, with special reference to the pho-

ton, is taken up next in Chapter 8. After this, we have a detailed introduction to quantum electrodynamics in Chapter 9, where we introduce the crucial concept of gauge invariance and give detailed derivations of important scattering processes at their lowest orders in perturbation theory.

Discrete symmetries can serve as a good guide in calculating higher order corrections. With this in mind, we discuss parity, time reversal, charge conjugation, and their combinations, in Chapter 10. Unlike most other textbooks on the subject which describe the $\mathcal{P}$, $\mathcal{T}$, $\mathcal{C}$ transformations only in the Pauli-Dirac representation of the Dirac matrices, we present them in a completely representation-independent way. Since other representations such as the Majorana or the chiral representations are very useful in some contexts, we hope that the general formulation will be of use to students and researchers.

We calculate loop diagrams in Chapter 11, showing how to use symmetries of a problem to parametrize quantum corrections. In this chapter, we restrict ourselves to finite contributions only. Some basic concepts of renormalization are then described, with detailed calculations, in Chapter 12. Only the electromagnetic gauge symmetry is used in these two chapters for illustrative purpose. But this leads to a more general discussion of symmetries, which is done in Chapter 13. This chapter also discusses the general ideas of symmetry breaking, and the related physics of Nambu-Goldstone theorem as well as its evasion through the Higgs mechanism. This is followed by an introduction to the Yang-Mills (or non-Abelian) gauge theories in Chapter 14. Finally, a basic introduction to standard electroweak theory and electroweak processes is given in Chapter 15. In keeping with our overall viewpoint, we discuss decays as well as scattering processes in this chapter.

We have tried to keep the book at the elementary level. In other words, this is a book for someone with no prior knowledge of the subject, and only a reasonable familiarity with special relativity and quantum mechanics. To set the stage as well as to help the reader, we visit briefly the relevant parts of classical field theory in Chapter 2. A short but comprehensive introduction to group theory also appears in Chapter 13, since we did not want to assume any background in group theory for the reader.

The most important tools to help the reader are the exercise problems in the book. These problems are not collected at the ends of chapters. Instead, any problem appears in the place of the text where we felt it would be most beneficial for the reader to have it worked out. Working it out at that stage should also prepare the reader for the ensuing parts of the chapter. Even if for some reason the reader does not want to work out a problem at that stage, we suggest strongly to at least read carefully the statement of the problem before proceeding further. Some of the problems come with notes or hints, some come with a relation that might be useful later. A few of the problems are marked with a $\star$ sign, implying that they

might be a little hard at that stage of the book, and the reader can leave it at that point to visit it later. A few have actually been worked out later in the book, but we have not marked them.

The book was submitted in a camera-ready form to the publishers. This means that we are responsible for all the mistakes in the book, including typographical ones. We have spared no effort to avoid errors, but if any has crept in, we would like to hear about it from the reader. We have set up a web site at

`http://tnp.saha.ernet.in/~pbpal/qftbk.html`

containing errata for the book, and a way of contacting us.

The book grew out of courses that both of us have taught at several universities and research institutes in India and abroad. We have benefited from the enquiries and criticisms of our students and colleagues. Indrajit Mitra went through the entire manuscript carefully and offered numerous suggestions. Many other friends and colleagues also read parts of the manuscript and made useful comments, in particular Kaushik Bhattacharya, Ed Copeland, A. Harindranath, H. S. Mani, José Nieves, Saurabh Rindani. Those who have taught and influenced us through their lectures, books and papers are too numerous to name separately. We thank them all.

We thank our respective institutes for extending various facilities while the book was being written.

Amitabha Lahiri
Palash B. Pal

April 2000

Notations

$\mu, \nu, \ldots$ Space-time indices of a vector or tensor.

$i, j, \ldots$ Spatial indices of a vector or tensor.

$g_{\mu\nu}$ (Components of) metric tensor, $\mathrm{diag}(1, -1, -1, -1)$.

p^μ contravariant 4-vector.

p_μ covariant 4-vector.

$\boldsymbol{p}$ 3-vector.

$\mathbf{p}$ Magnitude of the 3-vector $\boldsymbol{p}$, i.e., $|\boldsymbol{p}|$. We have used $\mathbf{p}^2$ and $\boldsymbol{p}^2$ interchangeably. The magnitude of the co-ordinate 3-vector has been denoted by r.

$\boldsymbol{a} \cdot \boldsymbol{b}$ Scalar product of 3-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a} \cdot \boldsymbol{b} = a^i b^i \equiv \sum a^i b^i$.

$a \cdot b$ Scalar product of 4-vectors a and b, $a \cdot b = a^\mu b_\mu \equiv \sum a^\mu b_\mu = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}$.

$\not{a}$ $\gamma^\mu a_\mu$.

$\mathscr{L}$ Lagrangian density, frequently called Lagrangian.

L Total Lagrangian ($= \int d^3x\, \mathscr{L}$).

$\mathscr{A}$ Action ($= \int dt\, L = \int d^4x\, \mathscr{L}$).

$\mathscr{H}$ Hamiltonian density, frequently called Hamiltonian.

H Total Hamiltonian ($= \int d^3x\, \mathscr{H}$).

$[A, B]_P$ Poisson bracket of A and B.

$[A, B]_-$ Commutator $AB - BA$.

$[A, B]_+$ Anticommutator $AB + BA$.

$\hat{a}(p), \hat{a}^\dagger(p)$ Annihilation and creation operators for antiparticles of particles created by $a^\dagger(p)$ and annihilated by $a(p)$, first encountered in §3.6.

$\Theta(x)$ Unit step function, defined in Eq. (3.13).

$:[\cdots]:$ Normal ordered product, first defined in §3.4.

$\mathscr{T}\,[\cdots]$ Time ordered product, first defined in §3.7.

A^{T} Transpose of the matrix A.

$A^{\dagger}$ For any matrix A, $A^{\dagger} \equiv \gamma_0 A^{\dagger} \gamma_0$.

σ^i Pauli matrices, given in Eq. (4.55).

τ^i Same as σ^i, but thought of as generators of some internal SU(2) symmetry.

$|e^-(\boldsymbol{p}, s)\rangle$ Electron state of 3-momentum $\boldsymbol{p}$ and spin s, first defined in §6.2.

$\Delta_F(p)$ Feynman propagator for scalar field in momentum space.

$S_F(p)$ Feynman propagator for fermion field in momentum space.

$D_{\mu\nu}(p)$ Feynman propagator for vector boson field in momentum space.

e Electric charge of proton. Electron carries charge $-e$.

α Fine structure constant, first defined in §1.5.

e^- Electron.

γ Photon.

S_{fi} S-matrix element between initial state $|i\rangle$ and final state $|f\rangle$.

$\mathscr{M}_{fi}$ Feynman amplitude between initial state $|i\rangle$ and final state $|f\rangle$.

$\overline{|\mathscr{M}|^2}$ Magnitude squared of Feynman amplitude after making spin and polarization sums and averages.

ϵ_r^μ Polarization vector for a vector boson.

$\mathcal{P}$ Parity transformation operator, defined in Ch. 10.

$\mathcal{C}$ Charge conjugation operator.

$\mathcal{T}$ Time reversal operator.

P Matrix of parity transformation acting on a fermion.

C Matrix of charge conjugation acting on a fermion.

T Matrix of time reversal acting on a fermion.

$\tilde{x}$ Parity transformed system of co-ordinates, $\tilde{x} = (t, -\boldsymbol{x})$.

Γ_μ General electromagnetic vertex, first defined in §11.1.

ε For dimensional regularization in N dimensions, $\varepsilon = 2 - \frac{1}{2}N$.

d_N Rank of γ-matrices in N-dimensional space-time, $\mathrm{tr}\,(\gamma_\mu \gamma_\nu) = d_N g_{\mu\nu}$.

T_a Generators of a Lie group.

f_{abc} Structure constants of a Lie group, $[T_a, T_b]_- = i f_{abc} T_c$.

D_μ Gauge covariant derivative $D_\mu = \partial_\mu + igT_a A^a_\mu$.

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Chapter 1

Preliminaries

The era of quantum mechanics began when Planck postulated that the modes of radiation of angular frequency ω could be counted as particles with energy $E = \hbar\omega$, and derived the law of blackbody radiation without running into infinities. The physical implications of this fundamental postulate became clearer when Einstein showed that quantization of radiation also explained the frequency dependence of photoelectric effect. This soon led to a proliferation of new ideas due to Bohr, Dirac, Born, Schrödinger, Heisenberg and many others, who applied the idea of quantization to particles and found remarkable success in describing subatomic phenomena. Dirac even constructed a relativistic theory of the electron, which was in excellent agreement with subatomic experiments. Despite its success, there were some essential shortcomings of quantum mechanics of particles.

1.1 Why Quantum Field Theory

One of these shortcomings was philosophical. Quantum mechanics started with the description of light in terms of photons, but the classical description of light was in terms of propagating electromagnetic fields. So a theory of photons required a prescription of how to quantize fields, a missing link between the two descriptions. As we shall see in the next several chapters, the structure required to quantize electromagnetic fields can be used to describe all elementary particles as quantum fields. Even the relativistic wave equation of Dirac was found to have the simplest description in terms of quantum fields

representing electrons and positrons.

Another problem of particle quantum mechanics is that it is valid in the non-relativistic regime by definition. This is not just because it uses non-relativistic Hamiltonians to solve various problems. In fact, the whole design of non-relativistic quantum mechanics defies relativity. For example, it uses the concept of potentials, which is untenable in any relativistic theory since it assumes the transfer of information at an infinite speed. Moreover, space and time are treated very differently in non-relativistic quantum mechanics. The spatial co-ordinates are operators, whereas time is a parameter, and we typically study the evolution of different operators, including the spatial co-ordinates, in time. Dirac equation, although covariant, treats space and time on different footings. In a truly relativistic theory, space and time should merge into a space-time, and one cannot make such fundamental distinction between the spatial part and the temporal part.

There is a practical problem as well. In nature there are processes in which new particles are created or annihilated. For example β-decay can be thought of as the decay of a neutron into a proton, an electron and an anti-neutrino:

$$n \rightarrow p + e^- + \bar{\nu}_e \,. \tag{1.1}$$

The neutron is annihilated in this process, while the proton, the electron etc are created. Quantum mechanics of particles deals with stable particles and their motions in various potentials. Without quantization of fields, any calculation involving the creation or annihilation of particles was essentially ad hoc if not downright wrong. Quantum Field Theory, by incorporating creation and annihilation of quanta or particles as its essential feature, allowed meaningful calculations of experimentally verifiable results.

When one goes to the relativistic regime, one encounters the possibility of creating new particles from energy. For example, if an electron and a positron collide with sufficient energy, the collision process can create extra electron-positron pairs:

$$e^- + e^+ \rightarrow e^- + e^+ + e^- + e^+ \,. \tag{1.2}$$

This of course can happen only if the total kinetic energy of the initial e^-e^+ pair is larger than the rest mass energy of the extra pair to be

created. To allow for such possibilities, one requires some mechanism to describe particle creation.

Even when the initial and the final states of a problem involve the same particles, a quantum process can go through intermediate states involving other particles which are created and destroyed in the process. Such intermediate states can contribute to the amplitude of the process and affect the result. It is important to consider these possibilities when one wants to make precision tests of quantum theory. The creation and annihilation of the intermediate particles in such cases can be handled only by going over to Quantum Field Theory.

1.2 Creation and annihilation operators

We now recall one problem in non-relativistic quantum mechanics which can be solved by the introduction of operators which create and annihilate quanta. This is the problem of the simple harmonic oscillator. Here we review this topic since it will be very useful for building up Quantum Field Theory in subsequent chapters.

The Hamiltonian of the one-dimensional oscillator is given by

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right) . \tag{1.3}$$

We define the following combinations of x and p:

$$a = \frac{1}{\sqrt{2m\hbar\omega}}\left(p - im\omega x\right) , \quad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(p + im\omega x\right) , \tag{1.4}$$

so that the Hamiltonian is

$$H = \frac{1}{2}\hbar\omega\left(a^\dagger a + aa^\dagger\right) . \tag{1.5}$$

If p and x were purely classical objects, we could have written $H = aa^\dagger\hbar\omega = a^\dagger a\hbar\omega$. But they are not. In Quantum Mechanics, they are operators which satisfy the commutation relation

$$[x, p]_- \equiv xp - px = i\hbar . \tag{1.6}$$

This implies that the objects a and $a^\dagger$, defined in Eq. (1.4), are also operators. Unlike x and p, these are non-hermitian operators, and they satisfy the relation

$$[a, a^\dagger]_- = 1 . \tag{1.7}$$

Moreover, they have the crucial property which can easily be checked directly from Eqs. (1.3) and (1.4):

$$[H, a]_- = -\hbar\omega a\,, \quad [H, a^\dagger]_- = \hbar\omega a^\dagger\,. \tag{1.8}$$

If we use the commutation relation of Eq. (1.7), we can rewrite the Hamiltonian as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\,. \tag{1.9}$$

The relations in Eq. (1.8) implies that, if $|n\rangle$ is an eigenstate of H with eigenvalue E_n, i.e., if

$$H\,|n\rangle = E_n\,|n\rangle\,, \tag{1.10}$$

then the states $a\,|n\rangle$ and $a^\dagger\,|n\rangle$ are also eigenstates:

$$Ha\,|n\rangle = (E_n - \hbar\omega)a\,|n\rangle\,, \quad Ha^\dagger\,|n\rangle = (E_n + \hbar\omega)a^\dagger\,|n\rangle\,. \tag{1.11}$$

In other words, the operator a seems to annihilate a quantum of energy, of amount $\hbar\omega$, from the state. On the other hand, $a^\dagger$ creates a quantum of energy. In this sense, they are the annihilation and the creation operators, respectively. Of course, in any physical process energy must be conserved, so the operators a or $a^\dagger$ cannot appear alone in the expression of any measurable quantity. For example, in the expression for the Hamiltonian they appear together, as we see in Eq. (1.9).

The ground state can be denoted by $|0\rangle$. Since this is the state of lowest energy, the annihilation operator a, acting on it, cannot produce a state of lower energy. Thus, this state must be totally annihilated by the operation of a:

$$a\,|0\rangle = 0\,. \tag{1.12}$$

Using Eq. (1.9) now, we can easily find its energy eigenvalue:

$$H\,|0\rangle = \frac{1}{2}\hbar\omega\,|0\rangle\,. \tag{1.13}$$

The first excited state, which we will denote by $|1\rangle$, can be defined by the state whose energy is larger than the ground state by one quantum, i.e.,

$$|1\rangle = a^\dagger\,|0\rangle\,. \tag{1.14}$$

The normalized n-th excited state is defined by

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle \ . \tag{1.15}$$

It is easy to see that the energy eigenvalue of the state $|n\rangle$ is given by

$$E_n = (n + \frac{1}{2})\hbar\omega \, . \tag{1.16}$$

□ **Exercise 1.1** If the ground state is normalized, i.e., if $\langle 0 \mid 0 \rangle = 1$, show that the excited states defined by Eq. (1.15) are also normalized.

□ **Exercise 1.2** The number operator is defined by

$$\mathcal{N} = a^\dagger a \, . \tag{1.17}$$

Show that its eigenvalue in the state given in Eq. (1.15) is n.

□ **Exercise 1.3** Construct the Hamiltonian for a system of N simple harmonic oscillators of frequencies ω_i, $i = 1, \ldots, N$.

Construct the normalized state in which the i-th oscillator is in the n_i-th excited state.

Construct the number operator and show that its eigenvalue in the above state is $\sum_i n_i$.

□ **Exercise 1.4** Suppose in a system there are operators which obey anticommutation relations $[a_r, a_s^\dagger]_+ \equiv a_r a_s^\dagger + a_s^\dagger a_r = \delta_{rs}$ and $[a_r, a_s]_+ = 0$, for $r, s = 1, \cdots, N$. Construct the generic normalized excited state. Show that no state can have two or more quanta of the same species.

1.3 Special relativity

Since our goal is to discuss relativistic field theories, here we give a quick summary of the mathematical structure of special relativity, which will help in setting up the notation.

In special relativity, the three spatial co-ordinates and time define the four components of the position 4-vector, which we will denote by x^μ. In keeping with established practice, we shall use Greek indices $\mu, \nu, \lambda, \ldots$ to denote components of a four-vector, and Latin indices $i, j, k, \ldots$ to denote its spatial components. In other words, Greek

indices take on values $0, 1, 2, 3$, while Latin indices take on values $1, 2, 3$. Thus,

$$x^\mu \equiv (x^0, x^i) \equiv (ct, \boldsymbol{x}), \tag{1.18}$$

where the factor of c is put in so that all components have dimensions of length. Here c is the speed of light in the vacuum, which is frame-independent according to one of the fundamental axioms of special relativity. The distance l between two points x and y in space-time can be written in terms of the Cartesian coordinates as

$$l^2 = (x^0 - y^0)^2 - \sum_{i=1}^{3} (x^i - y^i)^2. \tag{1.19}$$

We can write this in a compact form by defining the *metric tensor* $g_{\mu\nu}$ by

$$l^2 = g_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu). \tag{1.20}$$

Here, the values of the various components of the metric tensor are

$$g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1), \tag{1.21}$$

which means that $g_{00} = +1$, $g_{11} = g_{22} = g_{33} = -1$, and $g_{\mu\nu} = 0$ if $\mu \neq \nu$. In addition, we have used the *summation convention*, which says that any index which appears twice in the same term is summed over. We can use the metric to lower the indices, e.g.:

$$x_\mu = g_{\mu\nu} x^\nu = (ct, -\boldsymbol{x}). \tag{1.22}$$

Similarly, one can use the inverse of the matrix $g_{\mu\nu}$, which will be denoted by $g^{\mu\nu}$, to raise the indices:

$$x^\mu = g^{\mu\nu} x_\nu. \tag{1.23}$$

The inverse matrix must satisfy

$$g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda \equiv g^\mu{}_\lambda, \tag{1.24}$$

where δ^μ_λ denotes the Kronecker delta which is defined by

$$\delta^\mu_\lambda = \begin{cases} 1 \text{ if } \mu = \lambda \\ 0 \text{ if } \mu \neq \lambda. \end{cases} \tag{1.25}$$

Under a Lorentz transformation, the co-ordinates transform as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \,, \tag{1.26}$$

where the specific form of the matrix Λ is not very important for our purposes. These transformations are defined to be the ones which leave $x^\mu x_\mu$ invariant:

$$x'^\mu x'_\mu = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma \,. \tag{1.27}$$

Since this must equal $g_{\rho\sigma} x^\rho x^\sigma$, we obtain the relation

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \,, \tag{1.28}$$

which the Lorentz transformation matrices must satisfy.

□ **Exercise 1.5** Using the definition in Eq. (1.22), show that the Lorentz transformation rule for x_μ is given by

$$x_\mu \to x'_\mu = \Lambda_\mu{}^\nu x_\nu \,. \tag{1.29}$$

□ **Exercise 1.6** If a primed frame of reference moves with respect to an unprimed one with a uniform velocity v along the common x-axis, the Lorentz transformation equations between the co-ordinates in the two frames are given by

$$t' = \gamma(t - vx/c^2) \,, \quad x' = \gamma(x - vt) \,, \quad y' = y \,, \quad z' = z \,, \tag{1.30}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. Show that these transformations can be written in the form of Eq. (1.26) with $\Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu}$ for infinitesimal values of v, where $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Any object that remains invariant like $x^\mu x_\mu$ is a *scalar* quantity. Any four-component object that transforms like x^μ is called a *contravariant vector*, whereas a four-component object transforming like x_μ is a *covariant vector*.

The scalar product of two vectors A^μ and B^μ will be written as

$$\begin{aligned} g_{\mu\nu} A^\mu B^\nu &= A_\mu B^\mu = A^\mu B_\mu \\ &= A_0 B^0 + A_i B^i = A^0 B^0 - \boldsymbol{A} \cdot \boldsymbol{B} \,, \end{aligned} \tag{1.31}$$

where $\boldsymbol{A} \cdot \boldsymbol{B}$ is the usual three-dimensional scalar product. If $\phi(x)$ is a scalar function, so is a small change in it:

$$\delta\phi = \frac{\partial \phi}{\partial x^\mu} \delta x^\mu \,. \tag{1.32}$$

Since the right side is a scalar while δx^μ is a contravariant vector, $\frac{\partial \phi}{\delta x^\mu}$ must be a covariant vector, which will be denoted by $\partial_\mu \phi$, with the identification

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla} \right) . \tag{1.33}$$

For some arbitrary vector A^μ,

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial A^0}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{A} . \tag{1.34}$$

The wave operator $\Box$ is a scalar operator,

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 = \partial_\mu \partial^\mu . \tag{1.35}$$

The momentum 4-vector of a particle is given by

$$p^\mu = \left(\frac{E}{c}, \boldsymbol{p} \right) , \tag{1.36}$$

where E is the energy of the particle. The invariant scalar product $p^\mu p_\mu$ is denoted by

$$p^\mu p_\mu = m^2 c^2 , \tag{1.37}$$

where m is the rest-mass of the particle, which we will call simply *mass* for the sake of brevity.

□ **Exercise 1.7** Write Maxwell's equations and the Lorentz force formula in 4-vector notation. Get the signs right!

1.4 Space and time in relativistic quantum theory

In non-relativistic quantum mechanics, the status of time and of the spatial co-ordinates are very different, as we mentioned earlier. In a relativistic theory, they must be brought to par before proceeding. This is achieved by treating space on the same footing as time, i.e., by treating the spatial co-ordinates as parameters as well. Thus, the quantities that we can talk about in relativistic quantum theories are functions of both space and time, i.e., of space-time. Such functions

are called *fields* even in classical physics. For example, in classical physics, when we talk about the electromagnetic field, we talk about the electric field and the magnetic field which are functions of the spatial co-ordinates and time. Similarly, in Quantum Field Theory, we will talk about various kinds of fields, including the electromagnetic field. In classical or quantum mechanics, we could talk about objects which were functions of time alone. But the necessity to talk about space and time on the same footing automatically requires that we talk about fields.

It is important to realize that the uncertainty relations, which form the basis of a quantum theory, should be interpreted differently for this purpose. In non-relativistic quantum mechanics, the relation

$$\Delta x \, \Delta p_x \gtrsim \hbar \tag{1.38}$$

is interpreted in terms of the uncertainties in measurements of the co-ordinate x and the corresponding momentum p_x. On the other hand, the time-energy uncertainty relation,

$$\Delta t \, \Delta E \gtrsim \hbar \,, \tag{1.39}$$

is interpreted by saying that if one tries to make a measurement of energy to an accuracy ΔE, the measurement process should take a time Δt which is related to ΔE by the above relation. Now that x is also a parameter, the interpretation of Eq. (1.38) must also follow the line of that of Eq. (1.39). In other words, we should understand that the measurement of momentum to an accuracy Δp_x requires that the measurement be performed in a region whose spatial extent is larger than Δx given by Eq. (1.38).

1.5 Natural units

Elementary courses in physics start with three independent units, those of length, time and mass. As one proceeds, one introduces extra units, e.g., those of electric charge, temperature etc.

In discussions on Quantum Field Theory, it is customary to use a system of units in which there is only one fundamental unit, which we can take to be the unit of mass. The units of length and time are defined by declaring that, in this system of units,

$$\hbar = 1 \,, \quad c = 1 \,. \tag{1.40}$$

Table 1.1: Dimensions of various physical quantities in natural units. To go from energy units (MeV or GeV) to conventional units, we need to multiply by the conversion factor. For electric charge, 'conventional' will mean Heaviside-Lorentz unit for us.

Quantity	Dimension	Conversion factor
Mass	$[M]$	$1/c^2$
Length	$[M]^{-1}$	$\hbar c$
Time	$[M]^{-1}$	$\hbar$
Energy	$[M]$	1
Momentum	$[M]$	$1/c$
Force	$[M]^2$	$1/\hbar c$
Action	$[M]^0$	$\hbar$
Electric charge	$[M]^0$	$\sqrt{\hbar c}$

The resulting units are called *natural units*, which will be used in the rest of this book. The dimensions of various quantities in these units are presented in Table 1.1. We have given the conversion factors for quantities expressed in energy units, which is universally preferred in calculations. Readers should note that most other texts give conversion factors for mass units.

The electric charge is defined in such a way that Coulomb's law about the force $\boldsymbol{F}$ between two charges q_1 and q_2 takes the form

$$\boldsymbol{F} = \frac{q_1 q_2}{4\pi r^3}\,\boldsymbol{r}\,. \tag{1.41}$$

Thus electric charge is dimensionless, as shown in Table 1.1. The unit of electric charge will be taken to be the charge of the proton, e. The fine-structure constant α is related to this unit by the following relation:

$$\alpha = \frac{e^2}{4\pi}\,, \quad e > 0\,. \tag{1.42}$$

□ **Exercise 1.8** In natural units, the inverse lifetime of the muon is given by

$$\tau^{-1} = \frac{G_F^2 m^5}{192\pi^3}\,, \tag{1.43}$$

where m is the muon mass, 106 MeV. What is the dimension of G_F in natural units? Put in the factors of $\hbar$ and c so that the equation

can be interpreted in conventional units as well. From this, find the lifetime in seconds if $G_F = 1.166 \times 10^{-11}$ in MeV units. [Hint: Remember that MeV is a unit of energy.]

Chapter 2

Classical Field Theory

Before getting into the question of quantization, we discuss in this chapter some of the basic techniques involved in the covariant formulation of classical field theories.

2.1 A quick review of particle mechanics

2.1.1 Action principle and Euler-Lagrange equations

All theories of classical physics can be described by starting from a wonderful principle called the principle of least action. Suppose we are dealing with a non-relativistic system of particles and rigid bodies. We identify a set of coordinates $q_r(t)$ which, together with their time derivatives $\dot{q}_r(t)$, adequately describe the configuration and the evolution of the system. If we are clever enough, we can use our knowledge of the symmetries of the system to reduce this set to the smallest number of independent coordinates. On the other hand, in almost all the interesting cases it is useful not to make the reduction but instead choose a set of coordinates which have simple interrelations, making explicit the symmetries of the system.

We now define two mathematical objects called the Lagrangian L and the action $\mathscr{A}$, related by

$$\mathscr{A} = \int_{t_1}^{t_2} dt\; L(q_r(t), \dot{q}_r(t), t)\,. \tag{2.1}$$

The Lagrangian L is a function of the co-ordinates q_r and the velocities $\dot{q}_r$, and t_1 and t_2 are the initial and final times between which we are studying the evolution of the system. If there is a source or

sink of energy in the system under consideration, the Lagrangian can also depend explicitly on time.

The principle of least action then states that, among all trajectories that join $q_r(t_1)$ to $q_r(t_2)$, the system follows the one for which $\mathscr{A}$ is stationary.

To see what this implies, consider infinitesimal variations around a trajectory defined by $q_r(t)$, $\dot{q}_r(t)$:

$$\begin{aligned} q_r(t) &\to q_r(t) + \delta q_r(t)\,, \\ \dot{q}_r(t) &\to \dot{q}_r(t) + \frac{d}{dt}\delta q_r(t)\,. \end{aligned} \tag{2.2}$$

This introduces the following variation on the action defined in Eq. (2.1):

$$\delta\mathscr{A} = \int_{t_1}^{t_2} dt\, \left[\frac{\partial L}{\partial q_r}\delta q_r + \frac{\partial L}{\partial \dot{q}_r}\frac{d}{dt}\delta q_r\right], \tag{2.3}$$

where we have used the summation convention for the index r. We will assume that the system is conservative, so the Lagrangian does not have any explicit time-dependence. Integrating by parts, we can then write

$$\delta\mathscr{A} = \int_{t_1}^{t_2} dt\, \left[\frac{\partial L}{\partial q_r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r}\right]\delta q_r + \frac{\partial L}{\partial \dot{q}_r}\delta q_r\Big|_{t_1}^{t_2}. \tag{2.4}$$

Since we are looking at paths connecting given initial and final configurations $q_r(t_1)$ and $q_r(t_2)$, we shall not vary the end points. So we set

$$\delta q_r(t_1) = \delta q_r(t_2) = 0\,, \tag{2.5}$$

and the last term of Eq. (2.4) vanishes. The principle of least action states that, near the classical path, the variation of the action should be zero. Since this is true for any arbitrary variation of the path, i.e., for arbitrary $\delta q_r(t)$, $\delta\mathscr{A} = 0$ implies that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = \frac{\partial L}{\partial q_r}\,. \tag{2.6}$$

These are the equations of motion, also known as the *Euler-Lagrange equations*, corresponding to the co-ordinates q_r.

For a particle of mass m moving in a time-independent potential $V(\boldsymbol{x})$, a convenient choice for the Lagrangian is

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 - V(\boldsymbol{x})\,, \tag{2.7}$$

and the Euler-Lagrange equation as derived from this Lagrangian is

$$\frac{d}{dt}m\dot{\boldsymbol{x}} = -\boldsymbol{\nabla}V\,, \tag{2.8}$$

as expected from Newton's second law.

2.1.2 Hamiltonian formalism and Poisson brackets

The velocity-independent part of the Lagrangian can be thought of as generalized potential energy. So we can make the Euler-Lagrange equations look like Newton's second law by defining generalized momenta 'conjugate to q_r' as

$$p_r = \frac{\partial}{\partial \dot{q}_r} L(q, \dot{q})\,. \tag{2.9}$$

There will be one such equation for each value of the index r. Since the Lagrangian is a function of the positions and the velocities, the right hand sides of these equations are in general functions of the q's and the $\dot{q}$'s. Let us assume that these equations are invertible, i.e., the $\dot{q}$'s can be solved in terms of the positions and the momenta. Then the Hamiltonian is defined by the Legendre transformation

$$H(p, q) = p_r \dot{q}_r(p, q) - L\left(q, \dot{q}(p, q)\right)\,. \tag{2.10}$$

Indeed, by differentiation, we find

$$dH = \dot{q}_r dp_r + p_r d\dot{q}_r - \left(\frac{\partial L}{\partial q_r}dq_r + \frac{\partial L}{\partial \dot{q}_r}d\dot{q}_r\right)\,. \tag{2.11}$$

The terms with $d\dot{q}_r$ cancel each other owing to the definition of the momentum in Eq. (2.9). The term with dq_r can be simplified by the use of the Euler-Lagrange equation and Eq. (2.9), so that we can finally write

$$dH = \dot{q}_r dp_r - \dot{p}_r dq_r\,, \tag{2.12}$$

confirming the fact that the Hamiltonian is indeed a function of the positions and the momenta. Moreover, it directly leads to Hamilton's equations of motion:

$$\dot{q}_r = \frac{\partial H}{\partial p_r}\,, \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r}\,. \tag{2.13}$$

The Poisson bracket of any two dynamical variables is defined in the following way:

$$[f_1, f_2]_P = \frac{\partial f_1}{\partial q_r}\frac{\partial f_2}{\partial p_r} - \frac{\partial f_1}{\partial p_r}\frac{\partial f_2}{\partial q_r}\,. \tag{2.14}$$

Using this definition, it is then trivial to check that

$$[q_r, p_s]_P = \delta_{rs}\,, \tag{2.15}$$

and that the Hamilton's equations can be rewritten as

$$\dot{q}_r = [q_r, H]_P\,, \quad \dot{p}_r = [p_r, H]_P \tag{2.16}$$

More generally, if f is any dynamical variable, we can write

$$\dot{f} \equiv \frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]_P\,, \tag{2.17}$$

where $\frac{\partial f}{\partial t}$ occurs if f has an explicit time-dependence.

2.2 Euler-Lagrange equations in field theory

2.2.1 Action functional and Lagrangian

Going over to a field theory is trivial. Let us consider our system to be a field or a set of fields. A field is essentially a set of numbers at each point in space-time. We still want to describe this system by an action, and we still want to write the action as the time integral of a Lagrangian. But time is not the only independent variable in this case. We have to incorporate the information from every point in space into our action as well. We can do this by writing the Lagrangian as a spatial integral of some function of the fields. This has the added advantage of putting space and time on the same footing in the integral. The integrand has then the dimensions of Lagrangian density. However, in discussions of field theory, this is

always called the Lagrangian, and the word 'density' is implied for the sake of brevity. We will always use the word *Lagrangian* in this sense from now on. However, we will denote it by a different symbol, viz, $\mathscr{L}$. If, at any point, we need to use the word Lagrangian in the sense used in particle mechanics, we will call it the *total Lagrangian* and will denote it by L.

The Lagrangian should depend on the fields, which we denote generically by Φ^A. Here, different values of the index A can denote completely independent fields, or the different members of a set of fields which are related by some internal symmetry, or maybe the components of a field which transforms non-trivially under Lorentz transformations, e.g., like a vector. In any case, the field or fields denoted by Φ^A take the role of the co-ordinates q_r of Eq. (2.1).

What then replaces the velocities $\dot{q}_r$ of Eq. (2.1)? These should be the derivatives of the fields. However, we cannot have only the time derivative in this case, since spatial co-ordinates are independent parameters as well. We therefore should expect the Lagrangian to depend on the derivatives $\partial_\mu \Phi^A$.

And finally, since we are interested about fundamental fields, we should not expect any source or sink of energy or momentum in the system. Thus, the Lagrangian should not depend explicitly on the space-time co-ordinates. So finally we can write the action in the form

$$\mathscr{A} = \int_\Omega d^4x \, \mathscr{L}\left(\Phi^A(x), \partial_\mu \Phi^A(x)\right) , \tag{2.18}$$

where the region of integration Ω is the region of space-time we are interested in. For any physical experiment this region extends far enough away from the equipment and far enough into the past and future of the actual experiment so as to make the effect of all events outside that region negligible on the experiment. Usually Ω is taken to be the entire space-time for convenience.

A few things are worth noticing at this point. The action must be Poincaré invariant in a covariant theory, which means that it should be invariant under Lorentz transformations as well as space-time translations. Since the space-time volume element d^4x is Poincaré-invariant by itself, the Lagrangian as defined in Eq. (2.18) must also be invariant.

The Lagrangian density $\mathscr{L}$ is a function of x through its dependence on the fields Φ^A and their derivatives. The action on the other

hand is a number, i.e., it is a rule that associates a real number with a given configuration of the fields. If the field configuration is changed the number also changes in general. Such objects are called *functionals*. The action is a functional of the fields. The total Lagrangian L is a function of t but a functional of the fields at a given t.

2.2.2 Euler-Lagrange equations

Given the Lagrangian, the classical equations of motion can be derived from the principle of least action. This states that the system evolves through field configurations which keep the action $\mathscr{A}$ stationary against small variations of the fields Φ^A which vanish on the boundary of Ω.

Consider therefore the variations

$$\begin{aligned} \Phi^A(x) &\to \Phi^A(x) + \delta\Phi^A(x)\,, \\ \partial_\mu\Phi^A(x) &\to \partial_\mu\Phi^A(x) + \partial_\mu\delta\Phi^A(x)\,, \end{aligned} \tag{2.19}$$

such that $\delta\Phi^A$ vanishes on the boundary, which we denote by $\partial\Omega$. Using the functional dependence of the action denoted by Eq. (2.18), we can write

$$\begin{aligned} \delta\mathscr{A} &= \int_\Omega d^4x \left[\frac{\partial\mathscr{L}}{\partial\Phi^A}\,\delta\Phi^A + \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi^A\right)}\,\partial_\mu\delta\Phi^A\right] \\ &= \int_\Omega d^4x \left[\frac{\partial\mathscr{L}}{\partial\Phi^A} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi^A\right)}\right)\right]\,\delta\Phi^A \\ &\qquad + \int_\Omega d^4x\,\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Phi^A\right)}\,\delta\Phi^A\right] \end{aligned} \tag{2.20}$$

We now invoke Gauss theorem, which, for 4 dimensions, reads

$$\int_\Omega d^4x\,\partial_\mu F^\mu = \int_{\partial\Omega} dS_\mu\,F^\mu\,, \tag{2.21}$$

where F^μ is any well-behaved vector field, dS_μ is the outward pointing volume element on the boundary $\partial\Omega$. Using this, we can write the last term in Eq. (2.20) as an integral over $\partial\Omega$. However, the integrand now contains $\delta\Phi^A$, which vanishes everywhere on $\partial\Omega$. Thus, the last term in Eq. (2.20) vanishes.

As for the other terms, we notice that they must vanish for arbitrary variations of the fields. This can be true only if the expression multiplying $\delta\Phi^A$ vanishes, which implies

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial (\partial_\mu \Phi^A)} \right) = \frac{\partial \mathscr{L}}{\partial \Phi^A} \tag{2.22}$$

These equations, one for each Φ^A, are called the *Euler-Lagrange equations* for the system.

□ **Exercise 2.1** The Lagrangian (density) of an electromagnetic field interacting with a current j^μ is given by

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j_\mu A^\mu , \tag{2.23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Treating the A^μ's as the fields, find the Euler-Lagrange equations and show that they give the inhomogeneous Maxwell's equations.

□ **Exercise 2.2** For a real scalar field ϕ, the Lagrangian is $\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - V(\phi)$. Find the Euler-Lagrange equations.

□ **Exercise 2.3** Do the same for a complex scalar field ϕ with Lagrangian $\mathscr{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi)$.

It should be realized that the inverse problem cannot be solved. In other words, given the equation of motion, it is not possible to construct a unique Lagrangian. For example, consider two Lagrangians $\mathscr{L}$ and $\mathscr{L}'$ which differ by the divergence of some function of the fields:

$$\mathscr{L}' = \mathscr{L} + \partial_\mu K^\mu(\Phi) \tag{2.24}$$

When the fields are varied infinitesimally by $\delta\Phi^A$, the changes in actions defined from these two Lagrangians would differ by the amount

$$\delta\mathscr{A}' - \delta\mathscr{A} = \int d^4x \, \partial_\mu \left(\frac{\partial K^\mu}{\partial \Phi^A} \delta\Phi^A \right) \tag{2.25}$$

Using Gauss theorem of Eq. (2.21), this can be rewritten as an integral over the boundary $\partial\Omega$. But since $\delta\Phi^A$ vanishes on the boundary, we find that $\delta\mathscr{A}' = \delta\mathscr{A}$, which means that the equations of motion obtained from $\mathscr{L}'$ and $\mathscr{L}$ would be identical.

2.3 Hamiltonian formalism

The momenta canonical to the fields Φ^A are defined in analogy to systems with finite number of degrees of freedom:

$$\Pi_A \equiv \frac{\delta L}{\delta \dot{\Phi}^A} \,. \tag{2.26}$$

Here $\dot{\Phi}^A$ is the time-derivative of Φ^A. This is a *partial* derivative, since Φ^A depends also on the spatial co-ordinates. Moreover, L is a functional, so that the derivative indicated in Eq. (2.26) is really the derivative of a functional with respect to a function, i.e., a functional derivative. We will avoid using the partial derivative sign to remember this fact and will use the symbol δ instead.

In order to perform the functional differentiation in Eq. (2.26), we need to know the derivatives, with respect to $\dot{\Phi}$'s, of the fields Φ^A as well as of $\dot{\Phi}^A$ and $\boldsymbol{\nabla}\Phi^A$, since these are the building blocks of L. We take the cue from particle mechanics, where the generalized coordinates q_i and their corresponding velocities $\dot{q}_i$ are independent variables in the Lagrangian formalism. In particular, $\partial \dot{q}_i / \partial \dot{q}_j = \delta_{ij}$, whereas $\partial q_i / \partial \dot{q}_j = 0$. For fields, we replace the discrete index by the space co-ordinate $\boldsymbol{x}$ and the index A which characterize the field. So the fundamental functional derivatives are given by

$$\frac{\delta}{\delta \dot{\Phi}^A(t,\boldsymbol{x})} \dot{\Phi}^B(t,\boldsymbol{y}) = \delta^B_A \delta^3(\boldsymbol{x}-\boldsymbol{y}) \,, \tag{2.27}$$

whereas the functional derivatives of the fields and their spatial derivatives with respect to $\dot{\Phi}^A$ vanish identically. In Eq. (2.27), notice that the functional derivative of $\dot{\Phi}$ with respect to itself is not dimensionless, but has the dimensions of an inverse volume.

We now introduce the Hamiltonian of the system, which also should be understood to really mean the Hamiltonian density and will be denoted by the symbol $\mathscr{H}$. The volume integral of $\mathscr{H}$, which is usually called the Hamiltonian in the case of particle dynamics, will be called *total Hamiltonian* and denoted by H. The Hamiltonian $\mathscr{H}$ can be defined in terms of the canonical momenta in the following way:

$$\mathscr{H}(\Phi^A, \boldsymbol{\nabla}\Phi^A, \Pi_A) = \Pi_A \dot{\Phi}^A - \mathscr{L}(\Phi^A, \partial_\mu \Phi^A) \,. \tag{2.28}$$

Notice that the Lagrangian is a function of the fields and their partial derivatives with respect to all space-time co-ordinates. In the Hamiltonian, the field derivatives with respect to time have been replaced by the canonical momenta. However, the Hamiltonian can still be a function of the spatial derivatives. The total Hamiltonian is a functional in the same sense as the total Lagrangian.

□ **Exercise 2.4** Treating the A^μ's as the fields, find the canonical momenta associated with them, using the Lagrangian of Eq. (2.23). Can these equations, expressing the momenta in terms of A^μ's and their derivatives, be inverted to solve the $\dot{A}^\mu$'s? If not, can you think of a reason why?

□ **Exercise 2.5** There is some freedom in defining the electromagnetic potentials A^μ. Using this freedom, we can make $A^0 = 0$. This is an example of *gauge fixing*. In this gauge, treating the components of $\boldsymbol{A}$ as the independent fields, find out the momenta. Can these equations be inverted to solve the $\dot{A}^i$'s, where the index i runs over spatial components only?

The Poisson bracket of any two functionals F_1 and F_2 can be defined as

$$[F_1, F_2]_P = \int d^3\boldsymbol{x} \left(\frac{\delta F_1}{\delta \Phi^A(t,\boldsymbol{x})} \frac{\delta F_2}{\delta \Pi_A(t,\boldsymbol{x})} - \frac{\delta F_1}{\delta \Pi_A(t,\boldsymbol{x})} \frac{\delta F_2}{\delta \Phi^A(t,\boldsymbol{x})} \right) \tag{2.29}$$

These derivatives can be reduced to the following fundamental ones which are similar to Eq. (2.27):

$$\frac{\delta}{\delta \Phi^A(t,\boldsymbol{x})} \Phi^B(t,\boldsymbol{y}) = \delta^B_A \delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{2.30}$$

$$\frac{\delta}{\delta \Pi_A(t,\boldsymbol{x})} \Pi_B(t,\boldsymbol{y}) = \delta^A_B \delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{2.31}$$

whereas the functional derivatives of the fields with respect to the momenta, as well as vice versa, vanish. Thus, for example,

$$\begin{aligned} \left[\Phi^A(t,\boldsymbol{x}), \Pi_B(t,\boldsymbol{y})\right]_P &= \int d^3\boldsymbol{z} \left(\frac{\delta \Phi^A(t,\boldsymbol{x})}{\delta \Phi^C(t,\boldsymbol{z})} \frac{\delta \Pi_B(t,\boldsymbol{y})}{\delta \Pi_C(t,\boldsymbol{z})} \right. \\ &\qquad \left. - \frac{\delta \Phi^A(t,\boldsymbol{x})}{\delta \Pi_C(t,\boldsymbol{z})} \frac{\delta \Pi_B(t,\boldsymbol{y})}{\delta \Phi^C(t,\boldsymbol{z})} \right) \\ &= \int d^3\boldsymbol{z} \left(\delta^A_C \delta^3(\boldsymbol{x}-\boldsymbol{z}) \delta^C_B \delta^3(\boldsymbol{y}-\boldsymbol{z}) - 0 \right) \\ &= \delta^A_B \delta^3(\boldsymbol{x}-\boldsymbol{y}). \end{aligned} \tag{2.32}$$

□ **Exercise 2.6** Find the Hamiltonian for the complex scalar field whose Lagrangian has been given in Ex. 2.3 (p 18).

□ **Exercise 2.7** Consider a radiation field in a one-dimensional box or equivalently displacements on a string of length l with fixed ends. (This was the very first attempt at field theory.)

$$L = \int_0^l dx \left[\left(\frac{\partial u}{\partial t} \right)^2 - c^2 \left(\frac{\partial u}{\partial x} \right)^2 \right] \tag{2.33}$$

Let us express the field u in a Fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin\left(\frac{\omega_k x}{c} \right), \qquad \omega_k = k\pi c/l. \tag{2.34}$$

a) Calculate the 'momentum' p_k, canonically conjugate to q_k.

b) Write down the Hamiltonian as a function of q's and p's.

c) Take q_k and p_k to be operators with $[q_k, p_j] = i\hbar\delta_{kj}$. Define operators $a_k, a_k^\dagger$ by

$$q_k = \sqrt{\frac{\hbar}{2l\omega_k}} \left[a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t} \right] \tag{2.35}$$

Find the commutators of $a_k, a_k^\dagger$ with themselves and one another.

d) Calculate the Hamiltonian in terms of a's and $a^\dagger$'s.

2.4 Noether's theorem

Some Lagrangians are known from basic physics. But generally a system may not be as well understood, and we have to construct a Lagrangian that we expect to describe the system. In such constructions, it is often useful to rely on symmetry, i.e., transformations which leave the system invariant. It so happens that associated with each symmetry of a system is a conserved quantity. Thus if we know some conserved quantities of a system we can work backwards to find the symmetries of the system and from there make a guess at the form of a convenient Lagrangian. The result which relates symmetries to conserved quantities is known as Noether's theorem after its discoverer, while the conserved quantities are called Noether charges and Noether currents.

Let us consider infinitesimal transformations of the co-ordinate system

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu , \tag{2.36}$$

under which the fields transform as

$$\Phi^A(x) \to \Phi'^A(x') = \Phi^A(x) + \delta\Phi^A(x)\,. \tag{2.37}$$

The change in the action resulting from these transformations is given by

$$\begin{aligned}\delta\mathscr{A} = &\int_{\Omega'} d^4x'\ \mathscr{L}\left(\Phi'^A(x'), \partial'_\mu \Phi'^A(x')\right) \\ &- \int_{\Omega} d^4x\ \mathscr{L}\left(\Phi^A(x), \partial_\mu \Phi^A(x)\right)\,,\end{aligned} \tag{2.38}$$

where Ω' is the transform of Ω under the co-ordinate change. If $\delta\mathscr{A} = 0$, we say that the theory is invariant under the transformation.

In the first integral x' is a dummy variable, so we can replace it by x and write

$$\begin{aligned}\delta\mathscr{A} &= \int_{\Omega'} d^4x\ \mathscr{L}\left(\Phi'^A(x), \partial'_\mu \Phi'^A(x)\right) - \int_{\Omega} d^4x\ \mathscr{L}\left(\Phi^A(x), \partial_\mu \Phi^A(x)\right) \\ &= \int_{\Omega} d^4x\ \left[\mathscr{L}\left(\Phi'^A(x), \partial'_\mu \Phi'^A(x)\right) - \mathscr{L}\left(\Phi^A(x), \partial_\mu \Phi^A(x)\right)\right] \\ &\quad + \int_{\Omega'-\Omega} d^4x\ \mathscr{L}\left(\Phi'^A(x), \partial'_\mu \Phi'^A(x)\right)\end{aligned} \tag{2.39}$$

The last term is an integral over the infinitesimal volume $\Omega' - \Omega$, so we can replace it by an integral over the boundary $\partial\Omega$,

$$\begin{aligned}\int_{\Omega'-\Omega} d^4x\ \mathscr{L}\left(\Phi'^A, \partial'_\mu \Phi'^A\right) &= \int_{\partial\Omega} dS_\lambda\ \delta x^\lambda \mathscr{L}\left(\Phi^A, \partial_\mu \Phi^A\right) \\ &= \int_{\Omega} d^4x\ \partial_\lambda\left(\delta x^\lambda \mathscr{L}\left(\Phi^A, \partial_\mu \Phi^A\right)\right).\end{aligned} \tag{2.40}$$

Here we have suppressed the space-time index x as we no longer need to distinguish between x and x'. To get the first equation we have replaced Φ'^A by Φ^A as the difference is of higher order in δx^λ. In the second equation we have used Gauss theorem as shown in Eq. (2.21).

At this point, it is convenient to define the variation for fixed x, which gives the change in the functional forms only. For any function $f(x)$ whose functional form changes to $f'(x)$, we can write

$$\begin{aligned}\bar{\delta} f(x) &\equiv f'(x) - f(x) \\ &= [f'(x') - f(x)] - [f'(x') - f'(x)] \\ &= \delta f(x) - \partial_\mu f(x) \delta x^\mu\,,\end{aligned} \tag{2.41}$$

where we have again ignored terms of higher order in δx^μ in writing the last equality. Note that this variation $\bar{\delta}$ commutes with the partial derivatives ∂_μ as the variation is taken at the same space-time point.

This property of $\bar{\delta}$ allows us to simplify the integrals of Eq. (2.39) since we can write

$$\begin{aligned}\mathscr{L}\left(\Phi'^A(x), \partial'_\mu \Phi'^A(x)\right) &- \mathscr{L}\left(\Phi^A(x), \partial_\mu \Phi^A(x)\right) \\ &= \frac{\partial \mathscr{L}}{\partial \Phi^A}\,\bar{\delta}\Phi^A + \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \Phi^A\right)}\,\bar{\delta}\left(\partial_\mu \Phi^A\right) \\ &= \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \Phi^A\right)}\bar{\delta}\Phi^A\right), \qquad (2.42)\end{aligned}$$

using Euler-Lagrange equation, Eq. (2.22), to get the last equality. Putting this in Eq. (2.39) and using Eq. (2.40), we obtain

$$\begin{aligned}\delta \mathscr{A} &= \int_\Omega d^4x\, \partial_\mu\left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^A)}\,\bar{\delta}\Phi^A + \mathscr{L}\delta x^\mu\right] \\ &= \int_\Omega d^4x\, \partial_\mu\left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^A)}\,\delta\Phi^A - T^{\mu\nu}\delta x_\nu\right], \qquad (2.43)\end{aligned}$$

where we have used Eq. (2.41) again to bring back $\delta\Phi^A$, and defined

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^A)}\partial^\nu \Phi^A - g^{\mu\nu}\mathscr{L}, \qquad (2.44)$$

which is called the *stress-energy tensor*. We can now define a current

$$J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^A)}\,\delta\Phi^A - T^{\mu\nu}\delta x_\nu \qquad (2.45)$$

and write

$$\delta \mathscr{A} = \int_\Omega d^4x\, \partial_\mu J^\mu \qquad (2.46)$$

This relation holds for arbitrary variations of the fields and coordinates provided the equations of motion are satisfied.

Even though the right hand side of Eq. (2.46) is the integral of a total divergence, it does not necessarily vanish, because the current need not vanish on the boundary. If however there is a symmetry,

$\delta\mathscr{A}$ vanishes for arbitrary Ω even without the use of the equations of motion. In that case, we obtain a conservation law

$$\partial_\mu J^\mu = 0\,, \tag{2.47}$$

when the equations of motion are satisfied. This result is called *Noether's theorem*, and the current J^μ in this case is called the *Noether current*. Note that J^μ is defined only up to a constant factor.

It may so happen that under the symmetry transformations of Eqs. (2.36) and (2.37), the action does not remain invariant but instead changes by the integral of a total divergence,

$$\delta\mathscr{A} = \int_\Omega d^4x\, \partial_\mu Y^\mu \tag{2.48}$$

Then the expression for the corresponding conserved current will be the same as in Eq. (2.45), plus Y^μ.

It is easy to see that Eq. (2.47) implies the conservation of a charge, called *Noether charge*. It is defined by an integral over all space,

$$Q \equiv \int d^3x\, J^0\,, \tag{2.49}$$

provided the current vanishes sufficiently rapidly at spatial infinity. This is because

$$\frac{dQ}{dt} = \int d^3x\, \partial_0 J^0 = -\int d^3x\, \boldsymbol{\nabla}\cdot\boldsymbol{J} \tag{2.50}$$

The last expression can be converted into a surface integral on the spatial surface at infinity, on which it should vanish. Thus,

$$\frac{dQ}{dt} = 0\,, \tag{2.51}$$

i.e., the charge Q is conserved by the dynamics of the system. In what follows, we will find the specific form of the conserved current for various kinds of symmetries.

Space-time translations

The action constructed from relativistic fields is invariant under Lorentz transformations as well as translations of the co-ordinates.

Let us look at translations first. Suppose we make small changes in the space-time co-ordinates, defined by

$$x^\mu \to x^\mu + a^\mu \,, \tag{2.52}$$

where a^μ is constant, i.e., independent of x^μ. The fields do not change at any point, so

$$\delta\Phi^A = 0 \tag{2.53}$$

Since Eq. (2.47) is true for arbitrary a^μ, we conclude that

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.54}$$

This equation implies the conservation of the quantity

$$P^\mu = \int d^3x \; T^{0\mu} \,, \tag{2.55}$$

which is the 4-momentum of the field. Using the definition of the stress-energy tensor from Eq. (2.44), it is easy to see that P^0 is the total Hamiltonian defined through Eq. (2.28).

□ **Exercise 2.8** Find the stress-energy tensor for the following fields:

a) real scalar, as in Ex. 2.2 (p 18);
b) complex scalar, as in Ex. 2.3 (p 18);
c) electromagnetic field, as in Eq. (2.23) with $j^\mu = 0$.

Lorentz transformations

In this case the infinitesimal transformations are

$$x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu \,, \tag{2.56}$$

where the $\omega^{\mu\nu}$ are independent of x, and they are also antisymmetric because $x^\mu x_\mu$ is invariant. The field variations under this kind of operation are written in terms of a spin-matrix Σ by

$$\delta\Phi^A(x) = \frac{1}{2}\omega_{\lambda\rho}(\Sigma^{\lambda\rho})^A_B \Phi^B \tag{2.57}$$

The factor of $\frac{1}{2}$ appears because otherwise each independent component of ω is counted twice because of antisymmetry. Putting these in Eq. (2.45), we obtain

$$\partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^A)} \frac{1}{2}\omega_{\lambda\rho}(\Sigma^{\lambda\rho})^A_B \Phi^B - T^{\mu\lambda}\omega_{\lambda\rho}x^\rho \right] = 0 \tag{2.58}$$

Pulling out the $\omega^{\lambda\rho}$ which are constants, and also using the antisymmetry property of them, we can write this as

$$\partial_\mu \mathcal{M}^{\mu\lambda\rho} = 0\,, \tag{2.59}$$

where

$$\mathcal{M}^{\mu\lambda\rho} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^A)} (\Sigma^{\lambda\rho})^A_B \Phi^B - \left(T^{\mu\lambda} x^\rho - T^{\mu\rho} x^\lambda\right) \tag{2.60}$$

The conserved charges for this case are

$$J^{\lambda\rho} = \int d^3x\, \mathcal{M}^{0\lambda\rho} \tag{2.61}$$

The space components of this, J^{ij}, are related to angular momentum by

$$J^k = \frac{1}{2}\epsilon^{ijk} J_{ij}\,. \tag{2.62}$$

The quantities J^{0i} correspond to Lorentz boosts.

Internal symmetries

Internal symmetries are those which relate different fields at the same space-time point. In this case $\delta x^\mu = 0$, and let us write the infinitesimal transformations of the fields as

$$\Phi^A(x) \to \Phi^A(x) + f_r^A(x)\delta\epsilon_r, \qquad r = 1, \cdots, p\,, \tag{2.63}$$

which leaves the action invariant. Here the $\delta\epsilon_r$ are infinitesimal parameters independent of space-time, and $f_r^A(x)$ are specified functions of the fields Φ^A and their derivatives. The index r is not summed over, it indicates the type of symmetry. In other words, there may be several independent symmetries in a system. We can treat them separately by defining the conserved current for the r-th symmetry as

$$J_r^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi^A)} \frac{\delta \Phi^A}{\delta \epsilon_r}\,. \tag{2.64}$$

□ **Exercise 2.9** Consider the complex scalar field of Ex. 2.3 (p 18). The Lagrangian remains invariant under $\phi \to e^{-iq\theta}\phi$, $\phi^\dagger \to e^{iq\theta}\phi^\dagger$.

a) Use Noether's theorem to find the corresponding conserved current j^μ. Verify $\partial_\mu j^\mu = 0$.

b) Calculate the conserved Noether current for the same transformation when the electromagnetic field is also included, and the combined Lagrangian is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + [(\partial^\mu - iqA^\mu)\phi^\dagger][(\partial_\mu + iqA_\mu)\phi] - m^2\phi^\dagger\phi - V(\phi^\dagger\phi) \tag{2.65}$$

c) The Euler-Lagrange equations for A_μ should give the Maxwell equations: $\partial_\mu F^{\mu\nu} = j^\nu$. Show that the 'current' appearing in this equation is the same as the current obtained in part (b) above. [Note: Since the conserved current involves A^μ, the Lagrangian cannot be written like Eq. (2.23) for this case.]

□ **Exercise 2.10** Consider the Lagrangian of a real scalar field given in Ex. 2.2 (p 18), with $m = 0$ and $V(\phi) = \lambda\phi^4$. Verify that the action has the symmetry

$$x \to bx\,, \qquad \phi \to \phi/b\,. \tag{2.66}$$

Find the conserved current corresponding to this symmetry.

□ **Exercise 2.11** Suppose the action of a certain field theory is invariant under space-time translations as well as dilatations

$$x \to bx\,, \qquad \Phi \to \Phi\,. \tag{2.67}$$

Show that the stress-energy tensor in this case is traceless, i.e., $T^\mu{}_\mu = 0$.

Chapter 3

Quantization of scalar fields

Having outlined the necessary aspects of classical field theory in Ch. 2, we now turn to quantization of fields. In this chapter, we take up the quantization of the simplest kind of field which is described classically by one Lorentz invariant quantity at each space-time point. Such fields are called *scalar fields*. From now on, we will employ the natural units outlined in §1.5.

3.1 Equation of motion

For free particles of mass m, the 4-momentum satisfies the relation

$$p^\mu p_\mu - m^2 = 0\,. \tag{3.1}$$

In quantum theory, momentum is treated as an operator whose coordinate space representation is given by

$$p_\mu \to i\partial_\mu\,, \tag{3.2}$$

which operates on the wavefunctions. Thus, if we tried to represent the particles by a single wavefunction $\phi(x)$, we would simply substitute Eq. (3.2) into Eq. (3.1) and obtain

$$\left(\Box + m^2\right)\phi(x) = 0\,. \tag{3.3}$$

This is called the *Klein-Gordon equation* after the people who suggested it.

It looks simple enough, but there is a problem. The energy eigenvalues satisfy

$$E^2 = \boldsymbol{p}^2 + m^2 \quad \Rightarrow \quad E = \pm\sqrt{\boldsymbol{p}^2 + m^2}\,. \tag{3.4}$$

Notice that energy can be negative, and its magnitude can be arbitrarily large since the magnitude of $\boldsymbol{p}$, which we denote by $\mathbf{p}$, is not bounded. Thus, the system will have no ground state. The system will collapse into larger and larger negative values of energy and will be totally unstable. Hence, a wavefunction interpretation of the Klein-Gordon equation is not possible. We will see that this problem will be evaded once we interpret $\phi(x)$ as a quantum field, with a proper prescription for quantization.

3.2 The field and its canonical quantization

So we want to interpret $\phi(x)$ as a field. The classical equation of motion of this field will be given by Eq. (3.3). According to Ex. 2.2 (p 18), it can be derived from the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2 \tag{3.5}$$

□ **Exercise 3.1** Use the definition of the Lagrangian in Eq. (2.18) to find the mass dimensions of $\mathscr{L}$ in natural units, assuming the spacetime is N-dimensional. Use this to show that the mass dimensions of the field ϕ is $(N-2)/2$.

□ **Exercise 3.2** From the Lagrangian in Eq. (3.5), show that the momentum canonical to ϕ is

$$\Pi(x) = \dot{\phi}(x)\,, \tag{3.6}$$

and that the Hamiltonian is given by

$$\mathscr{H} = \frac{1}{2}\Big(\Pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\Big)\,. \tag{3.7}$$

In particle mechanics, the Poisson brackets between the coordinates and the canonical momenta were derived in Eq. (2.15). In order to go over to Quantum Mechanics, one can just use the replacement

$$[f_1, f_2]_P \to -i[f_1, f_2]_- \tag{3.8}$$

whereby one obtains the commutation relation of Eq. (1.6). Similarly, in the case of the fields, once we obtained the canonical momentum Π, we can use the same prescription to replace Eq. (2.32) by

$$[\phi(t,\boldsymbol{x}), \Pi(t,\boldsymbol{y})]_- = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{3.9}$$

This is called the *canonical commutation relation*, which implies that ϕ and Π can no more be considered as classical fields. They should now be treated as operators. The quantization procedure is a direct consequence of this commutation relation, as we will gradually see.

It might be a little dissatisfying to see that although we are trying to build up a relativistically covariant theory, the commutation relation of Eq. (3.9) does not seem to be a covariant equation. It seems that it somehow gives a preferential treatment to the time component and therefore cannot be valid in a general frame of reference. However, appearances are sometimes deceptive, and Eq. (3.9) can really be valid in any frame. The commutator is non-zero only if we consider the field ϕ and the canonical momentum Π at the same space-time point. If two space-time points coincide in one frame, they will coincide in any other frame. Thus, at least for this part, there is no difference if we go to any other frame. Next, consider the Lorentz transformation properties of the commutator bracket on the left hand side. Since ϕ is a scalar and $\Pi = \dot{\phi} = \partial_0\phi$, the non-trivial Lorentz transformation property of the left hand side comes only from ∂_0. In other words, the left hand side transforms like the time component of a 4-vector. As for the right hand side, notice that $d^4x\ \delta^3(\boldsymbol{x} - \boldsymbol{y}) = dt$ when integrated over a spatial region containing the point $\boldsymbol{y}$. By definition, dt transforms like the time component of a 4-vector. Since d^4x is a scalar, this means that $\delta^3(\boldsymbol{x} - \boldsymbol{y})$ also transforms like the time component of a 4-vector, and therefore Eq. (3.9) is covariant.

3.3 Fourier decomposition of the field

A classical field, being a function of space-time co-ordinates, can be Fourier decomposed. For the free field, we can write

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\ \delta(p^2 - m^2) A(p) e^{-ip\cdot x} \tag{3.10}$$

Here the factor outside the integral sign is just a normalizing factor, whose convenience will be seen later. Inside the integral sign, apart from the usual Fourier components $A(p)$, we have put a factor

$\delta(p^2 - m^2)$ to ensure that ϕ satisfies Eq. (3.3),

$$\left(\Box + m^2\right)\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\, (-p^2 + m^2)\delta(p^2 - m^2)A(p)e^{-ip\cdot x}. \\ = 0\,. \tag{3.11}$$

The δ-function, in association with the factor $(-p^2+m^2)$, guarantees that the integral will vanish.

Let us now assume that $\phi(x)$ is a hermitian operator, i.e., classically speaking the value of the field at any point is a real number. The generalization to complex scalar fields will be taken up in §3.6. The hermiticity condition, $\phi(x) = \phi^\dagger(x)$, then implies that the Fourier components satisfy the condition

$$A(-p) = A^\dagger(p)\,. \tag{3.12}$$

We now introduce the *step function* Θ, defined for any real variable z by

$$\Theta(z) = \begin{cases} 1 & \text{if } z > 0, \\ \frac{1}{2} & \text{if } z = 0, \\ 0 & \text{if } z < 0. \end{cases} \tag{3.13}$$

Obviously, $\Theta(p^0) + \Theta(-p^0) = 1$. Inserting this factor in Eq. (3.10) and changing the sign of the integrated variable p in the second term, we can write

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta(p^2 - m^2)\Theta(p^0) \left(A(p)e^{-ip\cdot x} + A^\dagger(p)e^{ip\cdot x}\right) \tag{3.14}$$

This form, then, involves only the states of positive energy.

In fact, the Fourier decomposition can be written in a somewhat simplified form if we eliminate p^0 from Eq. (3.14). We will use the result that given a function $f(z)$ which vanishes at the points z_n for $n = 1, 2, \ldots$,

$$\delta\left(f(z)\right) = \sum_n \frac{\delta(z - z_n)}{|df/dz|_{z=z_n}}\,, \tag{3.15}$$

provided the derivatives themselves do not vanish at the points z_n. Thus we can write

$$\delta(p^2 - m^2) = \delta\left((p^0)^2 - E_p^2\right) \\ = \frac{1}{2|p^0|}\left[\delta(p^0 - E_p) + \delta(p^0 + E_p)\right]\,, \tag{3.16}$$

where from now on, we reserve the symbol E_p for the positive energy eigenvalue:

$$E_p \equiv +\sqrt{\boldsymbol{p}^2 + m^2} \tag{3.17}$$

When Eq. (3.16) is substituted into Eq. (3.14), the second δ-function does not contribute to the integral since this is zero everywhere in the region where $\Theta(p^0) \neq 0$. The other δ-function contributes to the integral and gives

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right) \tag{3.18}$$

where

$$p^\mu = (E_p, \boldsymbol{p})\,, \tag{3.19}$$

and we have introduced the notation

$$a(p) = \frac{A(p)}{\sqrt{2E_p}}. \tag{3.20}$$

This is the form in which the Fourier decomposition will be most useful for us. The canonical momentum can also be written in a similar form involving integration over the 3-momentum only:

$$\Pi(x) = \dot{\phi}(x) = \int d^3p\, i\sqrt{\frac{E_p}{2(2\pi)^3}} \left(-a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right) \tag{3.21}$$

In the discussion about simple harmonic oscillators in §1.2, we started with the operators x and p, but switched to another set of operators, viz, a and $a^\dagger$. The situation is quite similar here. We started with $\phi(x)$ and $\Pi(x)$, but now we can do everything in terms of $a(p)$ and $a^\dagger(p)$. Just as the commutation relation between x and p in Eq. (1.6) led to the commutation relation of Eq. (1.7) between a and $a^\dagger$, here also we can use the canonical commutation relation of Eq. (3.9) to derive the commutation relation between the a's and the $a^\dagger$'s. For this, we need to invert the Fourier transform equations of Eqs. (3.18) and (3.21) to express $a(p)$ and $a^\dagger(p)$ in terms of $\phi(x)$ and $\Pi(x)$. From that, using Eq. (3.9), one can show that

$$\begin{aligned}
\left[a(p), a^\dagger(p')\right]_- &= \delta^3(\boldsymbol{p} - \boldsymbol{p}')\,, \\
\left[a(p), a(p')\right]_- &= 0\,, \\
\left[a^\dagger(p), a^\dagger(p')\right]_- &= 0
\end{aligned} \tag{3.22}$$

These commutation relations are reminiscent of the corresponding equations obtained for a linear harmonic oscillator in Eq. (1.7). If one takes a different prefactor in Eq. (3.10) than the one we used, the right hand side of the commutator of $a(p)$ and $a^\dagger(p)$ contains some extra factors.

We can derive Eq. (3.22) from the canonical commutation relation of Eq. (3.9), but we leave that for Ex. 3.4 (this page). Here, let us prove the converse. If we assume Eq. (3.22), using the Fourier decomposition of Eqs. (3.18) and (3.21), we can write

$$[\phi(t,\boldsymbol{x}),\Pi(t,\boldsymbol{y})]_- = \frac{i}{2(2\pi)^3}\int d^3p \int d^3p' \sqrt{\frac{E_{p'}}{E_p}} \times \left\{ \left[a(p),a^\dagger(p')\right]_- e^{-ip\cdot x+ip'\cdot y} - \left[a^\dagger(p),a(p')\right]_- e^{ip\cdot x-ip'\cdot y} \right\}, \tag{3.23}$$

where $x^0 = y^0 = t$. The commutators give δ-functions, and so the integration over $\boldsymbol{p}'$ can easily be performed. Note that once this is done, it forces $\boldsymbol{p} = \boldsymbol{p}'$, which also means $E_p = E_{p'}$. In the exponents, the time components in the dot products therefore cancel, and we are left with

$$[\phi(t,\boldsymbol{x}),\Pi(t,\boldsymbol{y})]_- = \frac{i}{2(2\pi)^3}\int d^3p \left\{ e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \right\} \tag{3.24}$$

This immediately reduces to Eq. (3.9) if we recall the integral representation of the δ-function of the form:

$$\delta^3(\boldsymbol{x}-\boldsymbol{y}) = \int \frac{d^3p}{(2\pi)^3} e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \tag{3.25}$$

□ **Exercise 3.3** Use the commutators of Eq. (3.22) to show that

$$[\phi(t,\boldsymbol{x}),\partial_i\phi(t,\boldsymbol{y})]_- = 0 \tag{3.26}$$

□ **Exercise 3.4** Write $a(p)$ and $a^\dagger(p)$ as inverse Fourier transforms of ϕ and Π, and derive the commutation relations of $a, a^\dagger$ from the canonical commutators of ϕ and Π.

We can now derive the total Hamiltonian of the system. Let us start with the volume integral of $\mathscr{H}$ given in Eq. (3.7), replace ϕ and Π by the expressions obtained in Eqs. (3.18) and (3.21) and perform

the integral over all space. Each of the exponential factors will give us a momentum δ-function. If we now integrate over one of the momenta, say p', we get for the different terms in the Hamiltonian the following results:

$$\int d^3x\, \Pi^2(x) = \int d^3p\, \frac{E_p}{2} \left[-a(p)a(-p)e^{-2iE_pt} + a(p)a^\dagger(p) + a^\dagger(p)a(p) - a^\dagger(p)a^\dagger(-p)e^{2iE_pt} \right] , \quad (3.27)$$

$$\int d^3x\, (\nabla\phi(x))^2 = \int d^3p\, \frac{\boldsymbol{p}^2}{2E_p} \left[a(p)a(-p)e^{-2iE_pt} + a(p)a^\dagger(p) + a^\dagger(p)a(p) + a^\dagger(p)a^\dagger(-p)e^{2iE_pt} \right] , \quad (3.28)$$

$$m^2 \int d^3x\, \phi^2(x) = \int d^3p\, \frac{m^2}{2E_p} \left[a(p)a(-p)e^{-2iE_pt} + a(p)a^\dagger(p) + a^\dagger(p)a(p) + a^\dagger(p)a^\dagger(-p)e^{2iE_pt} \right] . \quad (3.29)$$

This yields

$$H = \frac{1}{2} \int d^3p\, E_p \left[a^\dagger(p)a(p) + a(p)a^\dagger(p) \right] \quad (3.30)$$

Once again, the similarity with Eq. (1.5) is obvious. Moreover, one can check the following relations:

$$\begin{aligned} [H, a(p')]_- &= -E_{p'}a(p') \\ [H, a^\dagger(p')]_- &= +E_{p'}a^\dagger(p') , \end{aligned} \quad (3.31)$$

which are similar to the relations in Eq. (1.8). The analogy between the simple harmonic oscillator and the field is now complete. It is clear that we can interpret $a(p)$ as the annihilation operator and $a^\dagger(p)$ as the creation operator for a field quantum with momentum p. What was the positive energy component of the classical field now annihilates the quantum, and the negative energy component now creates the quantum. This quantum is what we call a particle of positive energy.

3.4 Ground state of the Hamiltonian and normal ordering

The ground state of the field, which is called the *vacuum* in field-theoretical parlance, can now be defined in analogy with Eq. (1.12).

Here, it reads

$$a(p)\,|0\rangle = 0 \quad \text{for all } p\,. \tag{3.32}$$

We will assume that this state is normalized, i.e.,

$$\langle 0\,|0\rangle = 1\,. \tag{3.33}$$

However, there is one problem here, which was not present in the case of the linear harmonic oscillator. To see this, let us use the commutation relations of Eq. (3.22) to rewrite the expression of Eq. (3.30) in the form

$$H = \int d^3p\; E_p \left[a^\dagger(p)a(p) + \frac{1}{2}\delta^3(0)_p\right]\,, \tag{3.34}$$

which would be the analog of Eq. (1.9) for this case. The subscript p after the δ-function in this equation is just to remind us that it is a δ-function for the zero of momentum, not the zero of co-ordinate. The distinction is not really relevant here, but the notation will be useful later. What comes out from this way of writing is that, if we consider the ground state energy, i.e., the expectation value of the total Hamiltonian in the ground state, we obtain

$$\langle 0\,|H|\,0\rangle = \frac{1}{2}\delta^3(0)_p \int d^3p\; E_p\,, \tag{3.35}$$

which is infinite!

The point is that, here we have one oscillator for each value of the momentum. Thus, we have an infinite number of oscillators. Since each oscillator has a finite ground state energy, the contributions from all such oscillators add up to give us an infinite result for the total ground state energy.

This, by itself, is not a catastrophe. After all, energy differences are physical quantities, absolute values of energies are not. We can just redefine the zero of energy such that the ground state energy vanishes. For this, we need a consistent prescription so that we do not run into trouble with other variables. This prescription is called *normal ordering*. Stated simply, it means that whenever we encounter a product of creation and annihilation operators, we define a normal-ordered product by moving all annihilation operators to the right of all creation operators *as if* the commutators were zero. Once

we have done the rearrangement, we treat them again as operators with the usual commutators. Using this algorithm on the expression of Eq. (3.30), we find that the normal-ordered total Hamiltonian, denoted by $:H:$, is given by

$$:H: = \int d^3p\, E_p\, a^\dagger(p) a(p)\,. \tag{3.36}$$

This expression immediately shows two things. First, for any state $|\Psi\rangle$,

$$\begin{aligned} \langle\Psi\,|\,:H:\,|\,\Psi\rangle &= \int d^3p\, E_p \left\langle \Psi \left| a^\dagger(p) a(p) \right| \Psi \right\rangle \\ &= \int d^3p\, E_p \left|\left| a(p)\,|\Psi\rangle \right|\right|^2 \end{aligned} \tag{3.37}$$

which is always non-negative. Second, using Eq. (3.32), we obtain

$$\langle 0\,|\,:H:\,|\,0\rangle = 0\,, \tag{3.38}$$

so that the vacuum defined earlier is really the state of lowest energy, i.e., the ground state.

□ **Exercise 3.5** a) For a real massive scalar field, calculate the 4-momentum $P^\mu = \int d^3x\, T^{0\mu}$ in terms of creation and annihilation operators. Show that normal ordering is not required for P^2.

b) Show that $[\phi, P_\mu]_- = i\partial_\mu\phi$.

3.5 Fock space

So far, we have defined only the vacuum state, which is a state with no particles. We shall need other states, in particular states with specified particle content, when we start describing physical events. We can define such states in analogy with excited states of oscillators. For example, we will define a one-particle state as

$$|p\rangle \equiv a^\dagger(p)\,|0\rangle \tag{3.39}$$

This state contains one quantum of the field ϕ with momentum $p^\mu = (E_p, \boldsymbol{p})$. Such states have positive norm, since

$$\langle p\,|p'\rangle = \delta^3(\boldsymbol{p} - \boldsymbol{p}')\,, \tag{3.40}$$

which comes from the commutation relations and the definition of the vacuum.

Similarly, we can define many-particle states. If a state has N particles with all different momenta $p_1, p_2, \ldots p_N$, it is defined by

$$|p_1, p_2, \ldots p_N\rangle \equiv a^\dagger(p_1)a^\dagger(p_2)\ldots a^\dagger(p_N)\,|0\rangle \tag{3.41}$$

On the other hand, if we want to construct a state with n particles of momentum p, it will be given by

$$|p(n)\rangle \equiv \frac{1}{\sqrt{n!}}\left(a^\dagger(p)\right)^n |0\rangle\ , \tag{3.42}$$

where the prefactor is needed for proper normalization.

Such multi-particle states distinguish field quantization (also called *second quantization*) from single-particle quantum mechanics (or *first quantization*). The vacuum, together with single particle states and all multi-particle states, constitute a vector space which is called the *Fock space*. The creation and annihilation operators act on this space.

□ **Exercise 3.6** The number operator in the Fock space can be defined as

$$\mathcal{N} = \int d^3p\, a^\dagger(p)a(p)\,. \tag{3.43}$$

Show that

$$\begin{aligned}[\mathcal{N}, a^\dagger(k)]_- &= a^\dagger(k)\,,\\ [\mathcal{N}, a(k)]_- &= -a(k)\,.\end{aligned} \tag{3.44}$$

Show that this operator correctly counts the number of particles in the states given in Eqs. (3.41) and (3.42).

3.6 Complex scalar field

3.6.1 Creation and annihilation operators

The Lagrangian for a complex scalar field $\phi(x)$ is given by

$$\mathcal{L} = (\partial^\mu\phi^\dagger)(\partial_\mu\phi) - m^2\phi^\dagger\phi\,. \tag{3.45}$$

Let us decompose the complex scalar field into its real and imaginary parts:

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \tag{3.46}$$

Then Eq. (3.45) can be rewritten as

$$\mathscr{L} = \sum_{A=1}^{2} \left[\frac{1}{2}(\partial^\mu \phi_A)(\partial_\mu \phi_A) - \frac{1}{2} m^2 \phi_A^2 \right] , \tag{3.47}$$

which is just the free Lagrangian of two independent real scalar fields ϕ_1 and ϕ_2. From this form, it is straightforward to calculate the momenta conjugate to ϕ_1 and ϕ_2, and impose the canonical commutation relations. We can perform Fourier decomposition of both ϕ_1 and ϕ_2 in the way outlined in §3.3, and define the annihilation and creation operators corresponding to each of them. The canonical commutation relations in this case imply

$$\left[a_1(p), a_1^\dagger(p') \right]_- = \left[a_2(p), a_2^\dagger(p') \right]_- = \delta^3(\boldsymbol{p} - \boldsymbol{p}') , \tag{3.48}$$

all other commutators being zero. To express these relations in terms of creation and annihilation operators associated with the complex field ϕ, let us define

$$\begin{aligned} a(p) &= \frac{1}{\sqrt{2}}(a_1(p) + i a_2(p)) , \\ \widehat{a}(p) &= \frac{1}{\sqrt{2}}(a_1(p) - i a_2(p)) , \end{aligned} \tag{3.49}$$

which mean

$$\begin{aligned} a^\dagger(p) &= \frac{1}{\sqrt{2}}(a_1^\dagger(p) - i a_2^\dagger(p)) , \\ \widehat{a}^\dagger(p) &= \frac{1}{\sqrt{2}}(a_1^\dagger(p) + i a_2^\dagger(p)) . \end{aligned} \tag{3.50}$$

We can now start with Eq. (3.18) for both ϕ_1 and ϕ_2 and use the definition of Eq. (3.46) to write the Fourier expansion of the field ϕ:

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(a(p) e^{-ip\cdot x} + \widehat{a}^\dagger(p) e^{ip\cdot x} \right) , \tag{3.51}$$

which also implies

$$\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left(\widehat{a}(p) e^{-ip\cdot x} + a^\dagger(p) e^{ip\cdot x} \right) \tag{3.52}$$

Moreover, the commutation relations of Eq. (3.48) can be cast into the form

$$\left[a(p), a^\dagger(p')\right]_- = \left[\hat{a}(p), \hat{a}^\dagger(p')\right]_- = \delta^3(\boldsymbol{p} - \boldsymbol{p}')\,, \tag{3.53}$$

whereas all other commutators vanish. Since $a_1^\dagger$ and $a_2^\dagger$ create the quanta of the fields ϕ_1 and ϕ_2 which are different, it is clear that there are two different particles in the theory. Since Eq. (3.53) presents the commutation relations in the form of Eq. (3.22), we can interpret $a^\dagger$ and $\hat{a}^\dagger$ to be the creation operators for these two particles. Of course, we could have chosen any two orthonormal combinations of a_1 and a_2, even a_1 and a_2 themselves, as the annihilation operators for our two particles. There is a special reason for quantizing ϕ and $\phi^\dagger$ in terms of a and $\hat{a}$, as we shall now see.

3.6.2 Particles and antiparticles

The Lagrangian of Eq. (3.45) is invariant under the transformation

$$\phi \to e^{-iq\theta}\phi\,, \quad \phi^\dagger \to e^{iq\theta}\phi^\dagger\,, \tag{3.54}$$

where θ does not depend on space-time. Following the procedure outlined in §2.4, we can calculate the Noether current for this invariance. For infinitesimal θ, we have

$$\delta\phi = -iq\theta\phi\,, \quad \delta\phi^\dagger = iq\theta\phi^\dagger\,. \tag{3.55}$$

So,

$$\begin{aligned} J^\mu &= \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}(-iq\phi) + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^\dagger)}(iq\phi^\dagger) \\ &= iq\left[(\partial^\mu\phi)\phi^\dagger - (\partial^\mu\phi^\dagger)\phi\right] \end{aligned} \tag{3.56}$$

The conserved charge Q is therefore given by

$$\begin{aligned} Q = \int d^3x\, J^0 &= \int d^3x\, iq\left[(\partial^0\phi)\phi^\dagger - (\partial^0\phi^\dagger)\phi\right] \\ &= q\int d^3p\, \left[a^\dagger(p)a(p) - \hat{a}^\dagger(p)\hat{a}(p)\right] \end{aligned} \tag{3.57}$$

The first term here, remembering the definition of Eq. (3.43), is just the number operator for the quanta created by $a^\dagger$. Let us denote it by N_a. Similarly, the second term is $N_{\hat{a}}$. Thus we obtain

$$Q = q\,(N_a - N_{\hat{a}}) \tag{3.58}$$

Notice the negative sign for the second term. If q is called the *charge* associated with each quanta created by $a^\dagger$, the first term is the total charge in such quanta. The quanta created by $\widehat{a}^\dagger$ then possess an opposite charge. These are called the *antiparticles* for the *particles* which are created by $a^\dagger$. Eq. (3.58) then tells us that the total charge in particles and antiparticles is conserved as a consequence of Noether's theorem.

□ **Exercise 3.7** Write Q in terms of $a_1(p), a_1^\dagger(p), a_2(p)$ and $a_2^\dagger(p)$ to verify that it cannot be written in terms of number operator of the fields ϕ_1 and ϕ_2.

Looking back at Eq. (3.51), we thus see that the operator $\phi(x)$ can do two things. The first term describes the annihilation of a particle, the second one the creation of an antiparticle. In either case, if $\phi(x)$ operates on any state, it will reduce the charge of the state by an amount q. Similarly, Eq. (3.52) implies that the operator $\phi^\dagger(x)$ can either create a particle, or annihilate an antiparticle. In other words, if it operates on a state, the charge will increase by q.

One may wonder at this point as to how the quantity q is defined. After all, in Eq. (3.54) where it first shows up, q is always multiplied by θ, so we can change the value of q by redefining θ with a multiplicative constant. But there is nothing mysterious about this. After all, any charge is defined only up to a multiplicative constant. For example, take the case of electric charge. If we multiply the values of the charges of all particles by a constant, the law of charge conservation will not be affected at all. The Maxwell's equations will also be unaffected, since the electromagnetic fields will be multiplied by the same constant. So, if we consider only one kind of particle in isolation, the value of the charge is really arbitrary. Only the ratios of charges of different particles have some physical meaning, and these ratios can be determined only when different kinds of particles are present and there is some interaction among them. Such instances will be encountered when we will discuss interactions in the later chapters.

3.6.3 Ground state and Hamiltonian

We can now construct the states of the Hamiltonian, following the procedure outlined in §3.4 and §3.5. The vacuum, or the ground

state, is defined in this case by the relation

$$a(p)\,|0\rangle = \hat{a}(p)\,|0\rangle = 0\,, \quad \text{for all } p\,. \tag{3.59}$$

In other words, the vacuum is the state which contains no particles and no antiparticles either. The state $|p\rangle$ defined in Eq. (3.39) will be a state containing one particle with energy E_p. Similarly, we can define a state

$$|\bar{p}\rangle \equiv \hat{a}^\dagger(p)\,|0\rangle\ , \tag{3.60}$$

which will contain one antiparticle with energy E_p. The states with more than one particles or antiparticles can be constructed similarly. The normal-ordered Hamiltonian can now be written as

$$:H: \,= \int d^3p\, E_p \left(a^\dagger(p)a(p) + \hat{a}^\dagger(p)\hat{a}(p)\right) \tag{3.61}$$

□ **Exercise 3.8** Find the Hamiltonian in terms of the field ϕ using the Lagrangian of Eq. (3.45). Use the Fourier decomposition of the field given in Eq. (3.51) and the prescription of normal ordering to express it in the form of Eq. (3.61).

3.7 Propagator

The goal of Quantum Field Theory is to describe particle interactions. Our interest will be in calculating cross-sections, transition probabilities, decay rates etc. For this, we need to know how particles move in space-time. We shall use Green's function techniques for this and define a propagator, which will be useful in later chapters.

Let us start with the Klein-Gordon equation with a source term:

$$\left(\Box + m^2\right)\phi(x) = J(x)\,. \tag{3.62}$$

Since the Klein-Gordon equation is the same whether the scalar field is real or complex, the discussion in this section is equally applicable for both cases.

In order to solve this equation, we first introduce a *propagator*, or Green's function, denoted by $G(x-x')$, which satisfies the equation

$$\left(\Box_x + m^2\right)G(x-x') = -\,\delta^4(x-x') \tag{3.63}$$

Here the notation $\Box_x$ indicates that the derivatives in the $\Box$ operator have to be taken with respect to the co-ordinates of x, not of x' Clearly, if this equation can be solved, we can represent the solution of Eq. (3.62) by

$$\phi(x) = \phi_0(x) - \int d^4x'\, G(x-x')J(x')\,, \tag{3.64}$$

where $\phi_0(x)$ is any solution of the free Klein-Gordon equation, Eq. (3.3). The boundary conditions are matched by choosing ϕ_0 appropriately.

Now, to solve for the propagator, we introduce its Fourier transform by the equation

$$G(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}G(p)\,, \tag{3.65}$$

where all the components of p^μ should be treated as independent variables, not related by the energy-momentum relation of Eq. (3.1). If we act on this equation by the operator appearing in Eq. (3.63), we obtain

$$\left(\Box_x + m^2\right) G(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}(-p^2+m^2)G(p). \tag{3.66}$$

Remembering that

$$\delta^4(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')}\,, \tag{3.67}$$

and comparing with Eq. (3.63), we obtain

$$G(p) = \frac{1}{p^2-m^2} = \frac{1}{(p^0)^2 - E_p^2}\,, \tag{3.68}$$

where E_p was defined in Eq. (3.17).

This expression, however, has an ambiguity. Let us put this expression into Eq. (3.65) and try to perform the integration over p^0. The integrand has poles at $p^0 = \pm E_p$, and therefore the integration cannot be performed over the real axis for p^0. This has nothing to do with quantum theory, since we have not used anything regarding quantization in this section so far. The problem lies even with the classical theory.

To get around this problem, we use a prescription by Feynman and complexify the propagator. Instead of the Green's function of Eq. (3.68), from now on we will use

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + i\varepsilon'} = \frac{1}{(p^0)^2 - (E_p - i\varepsilon)^2}\,, \tag{3.69}$$

where ε, ε' are infinitesimally small positive parameters, obviously related to each other by $\varepsilon' = 2E_p\varepsilon$. Of course ε has to be taken to zero at the end of calculations. For this reason, we will ignore the distinction between ε' and ε. Introduction of this parameter makes the propagator complex, and hence inherently non-classical.

Let us now rewrite Eq. (3.69) in the form

$$\Delta_F(p) = \frac{1}{2E_p}\left[\frac{1}{p^0 - (E_p - i\varepsilon)} - \frac{1}{p^0 + (E_p - i\varepsilon)}\right]. \tag{3.70}$$

This expression can be put into Eq. (3.65), which gives

$$\begin{aligned}\Delta_F(x - x') = \int \frac{d^3p}{(2\pi)^3}\,\frac{e^{+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}}{2E_p} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi}\, e^{-ip^0(t-t')} \\ \times \left[\frac{1}{(p^0 - E_p) + i\varepsilon} - \frac{1}{(p^0 + E_p) - i\varepsilon}\right]\end{aligned} \tag{3.71}$$

To perform the integrations over p^0, we need the following result from complex analysis:

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} d\zeta\, \frac{e^{-i\zeta t}}{\zeta + i\varepsilon} = -2\pi i\Theta(t)\,, \tag{3.72}$$

where the Θ-function was defined in Eq. (3.13). For the first term in Eq. (3.70), we can shift to the variable $p^{0'} = p^0 - E_p$ to perform the p^0-integration:

$$\int_{-\infty}^{+\infty} \frac{dp^{0'}}{2\pi}\, \frac{e^{-i(p^{0'}+E_p)(t-t')}}{p^{0'} + i\varepsilon} = -i\Theta(t - t')e^{-iE_p(t-t')} \tag{3.73}$$

Similarly, for the other term, we can substitute $p^{0'} = -p^0 - E_p$, which gives for this term

$$\int_{-\infty}^{+\infty} \frac{dp^{0'}}{2\pi}\, \frac{e^{-i(p^{0'}+E_p)(t'-t)}}{-p^{0'} - i\varepsilon} = i\Theta(t' - t)e^{-iE_p(t'-t)} \tag{3.74}$$

Combining the two terms then, we can write

$$\Delta_F(x-x') = -i\int \frac{d^3p}{(2\pi)^3 2E_p}\left[\Theta(t-t')e^{-iE_p(t-t')+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} + \Theta(t'-t)e^{iE_p(t-t')+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\right]. \quad (3.75)$$

Changing the sign of the integration variable $\boldsymbol{p}$ in the second term, this can be written as

$$i\Delta_F(x-x') = \int \frac{d^3p}{(2\pi)^3 2E_p}\left[\Theta(t-t')e^{-ip\cdot(x-x')} + \Theta(t'-t)e^{ip\cdot(x-x')}\right], \quad (3.76)$$

where now the components of the 4-vector p^μ are not independent, but are related by $p^0 = E_p$.

□ **Exercise 3.9** Show that $\Delta_F(x-x')$, given in Eq. (3.76), satisfies the equation

$$\left(\Box_x + m^2\right)\Delta_F(x-x') = -\,\delta^4(x-x')\,. \quad (3.77)$$

It is therefore a Green's function.

This form can be easily related to the quantized fields. For the sake of illustration, let us use the real scalar field. From Eq. (3.18), we can write

$$\begin{aligned}\phi(x)\,|0\rangle &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\, a^\dagger(p)e^{ip\cdot x}\,|0\rangle \\ &= \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\, e^{ip\cdot x}\,|p\rangle\,, \end{aligned} \quad (3.78)$$

where $|p\rangle$ is the one-particle state defined in Eq. (3.39). Notice that the annihilation operator in $\phi(x)$ does not contribute here owing to Eq. (3.32). On the other hand,

$$\langle 0|\,\phi(x') = \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}}\, e^{-ip'\cdot x'}\,\langle p'|\,, \quad (3.79)$$

where only the annihilation operator part of the Fourier decomposition of $\phi(x')$ is important. Using the normalization of Eq. (3.40), we thus obtain

$$\langle 0\,|\phi(x')\phi(x)|\,0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p}\, e^{ip\cdot(x-x')} \quad (3.80)$$

Thus, we can rewrite Eq. (3.76) in the following form using the field operators:

$$i\Delta_F(x-x') = \Theta(t-t')\langle 0|\phi(x)\phi(x')|0\rangle + \Theta(t'-t)\langle 0|\phi(x')\phi(x)|0\rangle , \tag{3.81}$$

which can be abbreviated to

$$i\Delta_F(x-x') = \langle 0|\mathscr{T}\left[\phi(x)\phi(x')\right]|0\rangle , \tag{3.82}$$

where $\mathscr{T}[\cdots]$ implies a *time-ordered product*, which means that the operator with the later time must be put to the left of the operator with the earlier time. In general, for any two scalar operators $A(x)$ and $B(x)$, the time-ordered product is defined as

$$\mathscr{T}\left[A(x)B(x')\right] \equiv \begin{cases} A(x)B(x') & \text{if } t > t', \\ B(x')A(x) & \text{if } t' > t. \end{cases} \tag{3.83}$$

Let us now see how the form given in Eq. (3.81) can be interpreted. First, suppose $t < t'$, in which case only the second term survives. From the expression in Eq. (3.78), we see that the operator $\phi(x)$ operating on the vacuum state to the right creates a particle at time t. On the other hand, for $\langle 0|\,\phi(x')$, as commented earlier, only the annihilation operator part contributes, so this part describes the annihilation of a particle at time t'. Thus the matrix element describes the amplitude of a particle being created at a time t, which propagates in space-time and is annihilated at a later time t'. Similarly, if $t > t'$, the first term of Eq. (3.81) describes the amplitude of a particle being created at a time t', which propagates in space-time and is annihilated at a later time t. This, in a sense, is the physical meaning of the propagator, which also shows why it should be important in calculating the amplitudes of various processes involving particle interactions. After all, between interactions, the particles just propagate in space-time! The mathematical formulation of these comments will be the subject of Ch. 6.

□ **Exercise 3.10** For the complex scalar field, show that Eq. (3.76) can be written as

$$i\Delta_F(x-x') = \langle 0|\mathscr{T}\left[\phi(x)\phi^\dagger(x')\right]|0\rangle \tag{3.84}$$

a) Argue that it represents the propagation of a particle from t' to t or the propagation of an antiparticle from t to t'.

b) Show that the propagator can also be written as

$$i\Delta_F(x-x') = \langle 0 \left| \mathscr{T} \left[\phi(x')\phi^\dagger(x)\right] \right| 0 \rangle \tag{3.85}$$

Describe this in terms of particle and antiparticle propagation.

Chapter 4

Quantization of Dirac fields

In Ch. 3, we dealt with the quantization of scalar fields. Such fields, by definition, are invariant under Lorentz transformations. In this chapter, we try something more complicated — fields which describe spin-$\frac{1}{2}$ particles.

4.1 Dirac Hamiltonian

The problem with the one-particle interpretation of the Klein-Gordon equation comes from the fact that we encounter negative energies for free particles. We showed how this problem is solved for scalar fields. Dirac tried an alternative solution to this problem, one that led to the correct description of spin-$\frac{1}{2}$ particles. He observed that this problem did not arise in non-relativistic quantum mechanics because the Schrödinger equation was linear in the time derivative. So he tried to construct a Hamiltonian which, in a sense, would represent the square root of the equation

$$H^2 = \boldsymbol{p}^2 + m^2 \,, \tag{4.1}$$

where H is the Hamiltonian operator and $\boldsymbol{p}$ is the operator for 3-momentum. In coordinate space these operators can be written as derivative operators, $H = i\frac{\partial}{\partial t}$, $\boldsymbol{p} = -i\boldsymbol{\nabla}$, as we know from their commutation properties. Dirac wanted an operator linear in the components of momentum which would square to $\boldsymbol{p}^2 + m^2$. But such an operator cannot be constructed using only numbers and functions as coefficients. So Dirac assumed that the square root of this equation

should be of the form

$$H = \gamma^0 (\boldsymbol{\gamma} \cdot \boldsymbol{p} + m) \,, \tag{4.2}$$

where γ^0, as well as $\boldsymbol{\gamma}$, are constant *matrices*. Since H and $\boldsymbol{p}$ are Hermitian operators, $\gamma^0\boldsymbol{\gamma}$ and γ^0 must be Hermitian matrices.

Let us now square both sides of Eq. (4.2). This gives

$$H^2 = \frac{1}{2}\left[\gamma^0\gamma^i, \gamma^0\gamma^j\right]_+ p^i p^j + \left[\gamma^0\gamma^i, \gamma^0\right]_+ p^i m + \left(\gamma^0\right)^2 m^2 \,, \tag{4.3}$$

where we have used the fact that the operators p^i and p^j commute, and introduced the anticommutator bracket:

$$[A, B]_+ \equiv [B, A]_+ = AB + BA \,. \tag{4.4}$$

Since this must agree with Eq. (4.1), we obtain the following relations for the γ-matrices:

$$\begin{aligned} \left[\gamma^0\gamma^i, \gamma^0\gamma^j\right]_+ &= 2\delta^{ij} \,, \\ \left[\gamma^0\gamma^i, \gamma^0\right]_+ &= 0 \,, \\ \left(\gamma^0\right)^2 &= 1 \,. \end{aligned} \tag{4.5}$$

Using the third relation, the second relation can be written as

$$\gamma^0\gamma^i\gamma^0 + \left(\gamma^0\right)^2\gamma^i = \gamma^0\gamma^i\gamma^0 + \gamma^i = 0 \,. \tag{4.6}$$

Multiplying on the left by γ^0, we now find that

$$\gamma^i\gamma^0 + \gamma^0\gamma^i = \left[\gamma^i, \gamma^0\right]_+ = 0 \,. \tag{4.7}$$

We can now use this to simplify the first of the trio,

$$\gamma^0\gamma^i\gamma^0\gamma^j + \gamma^0\gamma^j\gamma^0\gamma^i = -\gamma^i\gamma^j - \gamma^j\gamma^i = -\left[\gamma^i, \gamma^j\right]_+ = 2\delta^{ij} \,. \tag{4.8}$$

All the equations in Eq. (4.5) can therefore be expressed in a more compact form:

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} \,, \tag{4.9}$$

where $g^{\mu\nu}$ is the inverse metric, defined in Eq. (1.23). This is a matrix equation, where an identity matrix is implicit in the right hand side.

Dirac observed that this condition can be satisfied in four dimensions if the γ^μ's are at least 4×4 matrices. Despite appearances, the γ^μ do not transform as the components of a 4-vector. They are constant matrices. We will use them to construct vectors and tensors out of spin-$\frac{1}{2}$ fields.

The first property to note about the γ-matrices is that each of them is traceless:

$$\mathrm{Tr}\,\gamma^\mu = 0 \qquad \text{for all } \mu. \tag{4.10}$$

For example, $\mathrm{Tr}\,(\gamma^0) = -\mathrm{Tr}\,(\gamma^0\gamma^1\gamma^1)$ since $(\gamma^1)^2 = -1$ from Eq. (4.9). Since γ^0 and γ^1 anticommute, we can write the right hand side as $\mathrm{Tr}\,(\gamma^1\gamma^0\gamma^1)$. Using the cyclic property of the trace, $\mathrm{Tr}\,(ABC) = \mathrm{Tr}\,(CAB)$, this can be rewritten as $\mathrm{Tr}\,(\gamma^0\gamma^1\gamma^1) = -\mathrm{Tr}\,(\gamma^0)$, using the square of the matrix γ^1 once again. This shows that the trace of γ^0 vanishes. One can construct similar proofs for the γ^i as well.

Let us define some new matrices constructed from the γ-matrices. First we define

$$\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu]_- = -\sigma^{\nu\mu}\,. \tag{4.11}$$

There are six of these matrices in four dimensions because of the antisymmetry in the indices. They are related to the spin angular momentum of fields obeying the Dirac equation, or particles with half-integer spin.

Next we define the matrix

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\,, \tag{4.12}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor in four dimensions. The interchange of any two indices introduces a factor of (-1), and n interchanges introduce a factor of $(-1)^n$. It follows that $\epsilon_{\mu\nu\lambda\rho}$ has only one independent component, which is usually normalized to 1. We shall choose our convention by setting

$$\epsilon_{0123} = +1\,. \tag{4.13}$$

Some properties of the antisymmetric tensor are given in the Appendix. Note that for γ_5, we do not make any distinction between the upper and the lower indices.

Using γ^5, we can construct a set of sixteen linearly independent matrices

$$\Gamma = \left\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5\gamma^\mu, \gamma^5\right\} . \tag{4.14}$$

These can be thought of as a basis spanning a sixteen-dimensional space of matrices. Therefore the matrices must be at least 4×4.

□ **Exercise 4.1** Show that γ^5 is traceless, and that it satisfies

$$\left[\gamma^\mu, \gamma^5\right]_+ = 0 \quad \text{for all } \mu. \tag{4.15}$$

□ **Exercise 4.2** Show that the product of any two of the Γ_r can be expressed as a linear combination of all the Γ_r's.

□ **Exercise 4.3** ⋆ Show that the equation

$$\sum_{r=1}^{16} a_r \Gamma_r = 0 , \tag{4.16}$$

where a_r are complex numbers and Γ_r, $r = (1, \cdots, 16)$ are members of the set Γ defined above in Eq. (4.14), can be satisfied if and only if $a_r = 0$ for all r, i.e., the Γ_r are linearly independent. [Hint: Multiply the sum by each of Γ_r in turn and take the trace.]

Since γ^0, as well as the products $\gamma^0\gamma^i$, are Hermitian matrices, it follows that

$$\left(\gamma^i\right)^\dagger = \left(\gamma^i \left(\gamma^0\right)^2\right)^\dagger = \gamma^{0\dagger} \left(\gamma^i\gamma^0\right)^\dagger = \gamma^0\gamma^i\gamma^0 = -\gamma^i , \tag{4.17}$$

i.e., the matrices γ^i are anti-hermitian. We can include γ^0 and write a more compact relation,

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 . \tag{4.18}$$

This, along with the anticommutation relation of Eq. (4.9), are the basic properties which define the γ-matrices.

It is immediately clear that the specific form of the γ-matrices is not unique. Suppose we have a set of matrices $\{\gamma^\mu\}$ which satisfy both Eqs. (4.9) and (4.18). For any unitary matrix U, another set $\{\tilde{\gamma}^\mu\}$, defined by

$$\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger , \tag{4.19}$$

will also satisfy the same equations. The converse of this statement happens to be true as well. If two sets of matrices γ^μ and $\tilde{\gamma}^\mu$ satisfy Eq. (4.9), it can be shown that there exists a constant 4×4 matrix M such that

$$\tilde{\gamma}^\mu = M \gamma^\mu M^{-1} . \tag{4.20}$$

We will not prove this statement. But once it is assumed, it is easy to see that since in addition the matrices γ^μ and $\tilde{\gamma}^\mu$ both satisfy the hermiticity condition of Eq. (4.18), the matrix M has to be unitary.

We will not use any explicit representation of the γ-matrices since one representation is as good as any other. The only exception will be made in §4.3.2, where a specific form will be assumed just for the sake of constructing some explicit solutions.

4.2 Dirac equation

Using the Hamiltonian derived in the previous section, we can write the equation of motion of ψ as

$$i\frac{\partial}{\partial t}\psi(x) = \gamma^0 \left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\right)\psi(x) . \tag{4.21}$$

This is the celebrated Dirac equation. Multiplying from the left by γ^0, this can be put into the form

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) = 0 . \tag{4.22}$$

It is convenient (and conventional) to use the 'slash' notation — for any 4-vector a^μ, we will henceforth write

$$\not{a} \equiv \gamma^\mu a_\mu = \gamma_\mu a^\mu , \tag{4.23}$$

where $\gamma_\mu = g_{\mu\nu}\gamma^\nu$. In this notation, the Dirac equation takes the form

$$\left(i\not{\partial} - m\right)\psi(x) = 0 . \tag{4.24}$$

Since the γ^μ's are 4×4 matrices, it follows that $\psi(x)$ must be a column matrix with 4 entries. Since the γ-matrices are not unique, the solution $\psi(x)$ will also be non-unique. If $\tilde{\psi}(x)$ satisfies Eq. (4.22)

for a different choice of the γ-matrices defined in Eq. (4.19), it follows that

$$\tilde{\psi}(x) = U\psi(x)\,. \tag{4.25}$$

The Dirac equation should be relativistically covariant. This fact fixes the Lorentz transformation property of $\psi(x)$. To see this explicitly, let us assume that Eq. (4.22) is satisfied in a different frame of reference, and denote the objects in this frame with a prime. Thus,

$$\left(i\gamma^\mu \partial'_\mu - m\right)\psi'(x') = 0\,, \tag{4.26}$$

where

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu\,. \tag{4.27}$$

The relation between $\psi'(x')$ and $\psi(x)$ is assumed to be linear:

$$\psi'(x') = S(\Lambda)\psi(x)\,. \tag{4.28}$$

Putting this in Eq. (4.26) and using the transformation property of ∂_μ from Eq. (1.29), we obtain

$$i\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu S(\Lambda)\psi(x) - mS(\Lambda)\psi(x) = 0\,. \tag{4.29}$$

Multiplying this equation from the left by S^{-1} and comparing with Eq. (4.22), we conclude that $S(\Lambda)$ must satisfy the relation

$$S^{-1}(\Lambda)\gamma^\mu \Lambda_\mu{}^\nu S(\Lambda) = \gamma^\nu\,. \tag{4.30}$$

Let us now consider an infinitesimal Lorentz transformation, given by

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu}\,, \tag{4.31}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$, which follows from Eq. (1.28). Now, for infinitesimal transformations, we want $S(\Lambda)$ to be linear in the transformation parameters $\omega_{\mu\nu}$ and reduce to the identity transformation as $\omega_{\mu\nu} = 0$. Let us then write

$$S(\Lambda) = 1 - \frac{i}{4}\beta_{\mu\nu}\omega^{\mu\nu}\,, \tag{4.32}$$

where $\beta_{\mu\nu}$ are some 4×4 matrices, antisymmetric in their indices μ, ν. It follows that

$$S^{-1}(\Lambda) = 1 + \frac{i}{4}\beta_{\mu\nu}\omega^{\mu\nu} . \tag{4.33}$$

Putting these into Eq. (4.30) and keeping only first order terms in $\omega^{\mu\nu}$, we find

$$[\gamma_\mu, \beta_{\lambda\rho}]_- = 2i\left(g_{\mu\lambda}\gamma_\rho - g_{\mu\rho}\gamma_\lambda\right) , \tag{4.34}$$

a condition that the matrices $\beta_{\mu\nu}$ must satisfy. It is straightforward to check that this condition is satisfied by the $\sigma_{\mu\nu}$ matrices defined earlier,

$$\beta_{\mu\nu} = \sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]_- . \tag{4.35}$$

□ **Exercise 4.4** Using the γ-matrix algebra of Eq. (4.9), verify that the matrices defined in Eq. (4.35) do indeed satisfy Eq. (4.34). [Hint: First show that $[A, BC]_- = [A, B]_+ C - B[A, C]_+$.]

Thus for infinitesimal transformations, the Dirac equation is covariant if ψ transforms by the matrix given in Eq. (4.32), with the matrices $\beta_{\mu\nu}$ given by Eq. (4.35). Finite Lorentz transformations can be obtained by exponentiating Eq. (4.31). The corresponding transformation rule for ψ is

$$\psi'(x') = S(\Lambda)\psi(x) = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) . \tag{4.36}$$

Objects defined in space-time are classified as scalars, vectors, tensors, etc. according to their Lorentz transformation properties. Here we find a new class of objects — column vectors whose components mix among themselves according to Eq. (4.36). Such objects are called spinors (even if they do not satisfy the Dirac equation).

The angular momentum operators $J_{\mu\nu}$ generate the transformations

$$\psi'(x) = \left(1 - \frac{i}{2}J_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) , \tag{4.37}$$

corresponding to what we have called $\overline{\delta}$ in §2.4. Using the form of the infinitesimal Lorentz transformation matrix from Eq. (4.31), we can write

$$\psi'(x') = \psi'(x) + \omega^{\mu\nu}x_\nu\partial_\mu\psi'(x) . \tag{4.38}$$

Using the linearized forms of Eqs. (4.36) and (4.37) for infinitesimal $\omega^{\mu\nu}$, we obtain

$$J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}\sigma_{\mu\nu} \,. \tag{4.39}$$

The first term is the familiar expression for the orbital angular momentum, but there is also a second term. This shows that the solutions of Dirac equation carries some intrinsic angular momentum, or spin.

□ **Exercise 4.5** Show that under Lorentz transformations $\overline{\psi}\gamma^\mu\psi$ transforms as a vector and $\overline{\psi}\sigma^{\mu\nu}\psi$ as a tensor, where $\overline{\psi} \equiv \psi^\dagger\gamma^0$. [Note: This is the reason γ^μ carries a Lorentz index despite not being a vector.]

□ **Exercise 4.6** Suppose there exists a matrix C which has the property that

$$C^{-1}\gamma_\mu C = -\gamma_\mu^{\mathsf{T}} \quad \text{for all } \mu \,. \tag{4.40}$$

Show that the object

$$\psi_C \equiv C\gamma_0^{\mathsf{T}}\psi^* \tag{4.41}$$

has the same Lorentz transformation properties as ψ.

□ **Exercise 4.7** Using the anticommutation relation of the γ-matrices, prove the following contraction formulas:

$$\begin{aligned} \gamma_\lambda\gamma^\lambda &= 4 \,, \\ \gamma_\lambda\gamma_\mu\gamma^\lambda &= -2\gamma_\mu \,, \\ \gamma_\lambda\gamma_\mu\gamma_\nu\gamma^\lambda &= 4g_{\mu\nu} \,, \\ \gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma^\lambda &= -2\gamma_\rho\gamma_\nu\gamma_\mu \,. \end{aligned} \tag{4.42}$$

4.3 Plane wave solutions of Dirac equation

4.3.1 Positive and negative energy spinors

For a single particle interpretation, the $\psi(x)$ appearing in Eq. (4.22) would be a wave function. Although Dirac set forth to avoid the negative energy solutions, the irony of the situation is that the Dirac equation also has such solutions, just like the Klein-Gordon equation. To see that, let us assume that we are trying a plane wave solution in the rest frame of a particle. The time dependence of this

solution would be of the form e^{-iEt}. From Eq. (4.22) we see that the eigenvalue E should satisfy the equation

$$\left(\gamma^0 E - m\right)\psi(x) = 0\,. \tag{4.43}$$

However, $(\gamma^0)^2 = 1$, as we have seen, so the eigenvalues of the matrix γ^0 can be $+1$ or -1. Thus we obtain the solution of the eigenvalue equation to be

$$E = \pm m\,. \tag{4.44}$$

There will be two independent eigenvectors for each sign of the energy, as should be expected since ψ, being a 4-component object, can have four independent solutions.

Thus both positive and negative eigenvalues will occur in a single particle interpretation of the Dirac equation. For a general value of the 3-momentum $\boldsymbol{p}$, let us represent solutions in the form

$$\psi(x) \sim \begin{cases} u_s(\boldsymbol{p})e^{-ip\cdot x}\,, \\ v_s(\boldsymbol{p})e^{+ip\cdot x}\,, \end{cases} \tag{4.45}$$

up to normalization factors, where the index s labels the two independent solutions of each kind. The positive energy u-solutions correspond to a particle with momentum $\boldsymbol{p}$ and energy $E_p = +\sqrt{\boldsymbol{p}^2 + m^2}$, whereas the negative energy v-solutions correspond to a particle with momentum $-\boldsymbol{p}$ and energy $-E_p = -\sqrt{\boldsymbol{p}^2 + m^2}$.

Putting Eq. (4.45) into Eq. (4.22), we can see that the spinors satisfy the relations

$$\begin{aligned} (\not{p} - m)\, u_s(\boldsymbol{p}) &= 0\,, \\ (\not{p} + m)\, v_s(\boldsymbol{p}) &= 0\,. \end{aligned} \tag{4.46}$$

For spinors, hermitian conjugation turns out to be less useful than what we may call Dirac conjugation, defined by

$$\overline{\psi} \equiv \psi^\dagger \gamma^0\,. \tag{4.47}$$

Using this definition and Eq. (4.18), it is easy to see that Eq. (4.46) can also be written as

$$\begin{aligned} \overline{u}_s(\boldsymbol{p})\,(\not{p} - m) &= 0\,, \\ \overline{v}_s(\boldsymbol{p})\,(\not{p} + m) &= 0\,. \end{aligned} \tag{4.48}$$

The normalization of the spinors has not been specified yet. Let us now fix them with the conditions

$$u_r^\dagger(\boldsymbol{p})u_s(\boldsymbol{p}) = v_r^\dagger(\boldsymbol{p})v_s(\boldsymbol{p}) = 2E_p\delta_{rs}\,,$$
$$v_r^\dagger(\boldsymbol{p})u_s(-\boldsymbol{p}) = u_r^\dagger(\boldsymbol{p})v_s(-\boldsymbol{p}) = 0\,. \tag{4.49}$$

□ **Exercise 4.8** Multiply the u-equation of Eq. (4.46) from the left by $\overline{u}\gamma^\mu$ and its conjugate equation, Eq. (4.48), by $\gamma^\mu u$ from the right. Add the resulting equations to show that

$$m\overline{u}(\boldsymbol{p})\gamma^\mu u(\boldsymbol{p}) = p^\mu \overline{u}(\boldsymbol{p})u(\boldsymbol{p})\,. \tag{4.50}$$

Derive a similar equation for the v-spinors. In particular, putting $\mu = 0$, show that the normalization equations can also be written in the form

$$\overline{u}_r(\boldsymbol{p})u_s(\boldsymbol{p}) = -\,\overline{v}_r(\boldsymbol{p})v_s(\boldsymbol{p}) = 2m\delta_{rs}\,. \tag{4.51}$$

Note that this is an incorrect normalization for $m = 0$, while Eq. (4.49) holds for all m.

□ **Exercise 4.9** Use the same technique to show a more general identity, where the spinors on the two sides correspond to different momenta. This leads to

$$\overline{u}(\boldsymbol{p}')\,\gamma^\mu\,u(\boldsymbol{p}) = \frac{1}{2m}\,\overline{u}(\boldsymbol{p}')\,[(p+p')^\mu - i\sigma^{\mu\nu}q_\nu]\,u(\boldsymbol{p})\,, \tag{4.52}$$

where $q = p - p'$. This is called the *Gordon identity*. Derive a similar equation for the v-spinors.

□ **Exercise 4.10** ⋆ Since $u(\boldsymbol{p})$ is a column matrix and $\overline{u}(\boldsymbol{p})$ is a row matrix, the product $u\overline{u}$ must be a 4×4 square matrix, and so is $v\overline{v}$. Show that, if we sum over spins, we obtain

$$\sum_s u_s(\boldsymbol{p})\overline{u}_s(\boldsymbol{p}) = \not{p} + m\,,$$
$$\sum_s v_s(\boldsymbol{p})\overline{v}_s(\boldsymbol{p}) = \not{p} - m\,. \tag{4.53}$$

[Hint: Take, for example, the first equation. Apply the matrices on both sides on the four basis spinors and show that the results are the same. If both sides act similarly on all the basis spinors, they must act the same way on any spinor and therefore must be the same matrix.]

4.3.2 Explicit solutions in Dirac-Pauli representation

So far, we have done everything in a representation-independent way. As we remarked earlier, all representations are equivalent, and for

most purposes we will use the representation-independent properties of the γ-matrices. However, it is useful to derive the plane wave solutions in a particular representation in order to gain some insight. For this, we take the Dirac-Pauli representation, which is given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \tag{4.54}$$

where I is the 2×2 unit matrix and the σ^i's are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.55}$$

To find the solutions, we put the representation of the γ-matrices into Eq. (4.46). First, note that

$$\not{p} - m = \gamma_0 E_p - \boldsymbol{\gamma} \cdot \boldsymbol{p} - m = \begin{pmatrix} E_p - m & -\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & -E_p - m \end{pmatrix} , \tag{4.56}$$

where once again each entry of this matrix is really a 2×2 matrix, as in Eq. (4.54). Thus, if we write the u-spinors also as

$$u \equiv \begin{pmatrix} \phi_t \\ \phi_b \end{pmatrix} , \tag{4.57}$$

where ϕ_t and ϕ_b are the top two and the bottom two components respectively, we obtain the following equations for them using Eq. (4.46):

$$\begin{aligned} (E_p - m)\phi_t - \boldsymbol{\sigma} \cdot \boldsymbol{p}\phi_b &= 0 \\ \boldsymbol{\sigma} \cdot \boldsymbol{p}\phi_t - (E_p + m)\phi_b &= 0 . \end{aligned} \tag{4.58}$$

These are two homogeneous equations with two unknowns, and solutions exist since the determinant of the co-efficients is zero, which can be seen by explicit calculation.

□ **Exercise 4.11** Verify the statement above, that the determinant of the co-efficients of Eq. (4.58) is zero.

We can now use any one of the equations in Eq. (4.58) to relate ϕ_t and ϕ_b. Using the second one, we get

$$\phi_b = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E_p + m} \phi_t . \tag{4.59}$$

Thus, there can be two independent solutions since ϕ_t is a two-component column vector. Let us choose the independent solutions for ϕ_t to be proportional to

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.60}$$

The elements of χ_+ and χ_- are numbers, not matrices. We can now write down the solutions for the u-spinors:

$$u_\pm(\boldsymbol{p}) = \sqrt{E_p + m} \begin{pmatrix} \chi_\pm \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E_p + m} \chi_\pm \end{pmatrix}, \tag{4.61}$$

where the prefactor has been put in so that our solutions comply with the normalization condition of Eq. (4.49).

Similarly, we can find the solutions for the v-spinors, which are

$$v_\pm(\boldsymbol{p}) = \pm\sqrt{E_p + m} \begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E_p + m} \chi_\mp \\ \chi_\mp \end{pmatrix}. \tag{4.62}$$

These solutions also have been normalized according to Eq. (4.49). The mismatch of subscripts on the two sides is a conventional choice, which we will discuss again in §4.6.

□ **Exercise 4.12** Another way of finding the explicit solutions is to solve first in the rest frame, where $p^\mu = (m, \mathbf{0})$. In this frame, we can choose the independent solutions to be of the following form:

$$u_+(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_-(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$v_+(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{2m} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{4.63}$$

It is easy to check that these satisfy the normalization conditions given above. Find the correctly normalized solutions for any 3-momentum by noting that $(\not{p} + m)u_\pm(\mathbf{0})$ and $(\not{p} - m)v_\pm(\mathbf{0})$ are solutions of Eq. (4.46).

□ **Exercise 4.13** Find the matrix γ_5 in the Dirac-Pauli representation.

4.4 Projection operators

Since we have multiple solutions, a useful concept is that of the projection operators, which project out the appropriate solutions in some particular situation. We discuss some such projection operators, which will be quite useful in later parts of the book.

4.4.1 Projection operators for positive and negative energy states

For example, if we are interested in states of either positive or negative energy, but not both, we can use the energy projection operators

$$\Lambda_{\pm}(\boldsymbol{p}) = \frac{\pm\not{p} + m}{2m} \,. \tag{4.64}$$

For any two vectors a and b, the anticommutation relation of γ-matrices imply the relation

$$\not{a}\not{b} = 2a \cdot b - \not{b}\not{a} \,. \tag{4.65}$$

In particular, when $a = b$, this implies

$$\not{a}\not{a} = a^2 \,. \tag{4.66}$$

It follows that acting on any solution of the Dirac equation, $\Lambda_{\pm}$ satisfy

$$\Lambda_{\pm}^2 = \Lambda_{\pm}, \quad \Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0, \quad \Lambda_+ + \Lambda_- = 1 \,. \tag{4.67}$$

Using Eq. (4.46), we see that the plane wave solutions of positive and negative energy which we found are eigenstates of these energy projection operators,

$$\begin{aligned} \Lambda_+(\boldsymbol{p})u_s(\boldsymbol{p}) &= u_s(\boldsymbol{p}) \,, \\ \Lambda_-(\boldsymbol{p})v_s(\boldsymbol{p}) &= v_s(\boldsymbol{p}) \,. \end{aligned} \tag{4.68}$$

On the other hand

$$\Lambda_+(\boldsymbol{p})v_s(\boldsymbol{p}) = \Lambda_-(\boldsymbol{p})u_s(\boldsymbol{p}) = 0 \,. \tag{4.69}$$

Similarly, using Eq. (4.48) we can write

$$\begin{aligned} \overline{u}_s(\boldsymbol{p})\Lambda_+(\boldsymbol{p}) &= \overline{u}_s(\boldsymbol{p}) \,, \\ \overline{v}_s(\boldsymbol{p})\Lambda_-(\boldsymbol{p}) &= \overline{v}_s(\boldsymbol{p}) \,, \end{aligned} \tag{4.70}$$

as well as

$$\bar{v}_s(\boldsymbol{p})\Lambda_+(\boldsymbol{p}) = \bar{u}_s(\boldsymbol{p})\Lambda_-(\boldsymbol{p}) = 0\,. \tag{4.71}$$

4.4.2 Helicity projection operators

Another useful pair are the helicity projection operators. The spin of a Dirac fermion, when measured along any direction, can be either + or − in units of $\frac{1}{2}\hbar$. A special direction is the direction of motion of the fermion (except in its rest frame, of course). The spin measured along the direction of motion is called *helicity*, and the helicity projection operators project out states of positive and negative helicities.

We have noted in Eq. (4.39) that $\frac{1}{2}\sigma_{\mu\nu}$ is the spin operator for fermions. Let us therefore define a 3-vector

$$\boldsymbol{\Sigma} \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12})\,. \tag{4.72}$$

Next we project this operator along the direction of motion of the fermion,

$$\boldsymbol{\Sigma}_{\boldsymbol{p}} = \frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{\mathbf{p}}\,. \tag{4.73}$$

This has eigenvalues $+1$ and -1.

□ **Exercise 4.14** Check the eigenvalues of $\boldsymbol{\Sigma}_{\boldsymbol{p}}$. (You can do this by going to the Dirac-Pauli representation, but also try to do it in a representation-independent way.)

We can now define the helicity projection operators,

$$\Pi_\pm(\boldsymbol{p}) = \frac{1}{2}\left(1 \pm \boldsymbol{\Sigma}_{\boldsymbol{p}}\right)\,. \tag{4.74}$$

These operators satisfy the relations expected of projection operators,

$$\Pi_\pm^2 = \Pi_\pm, \quad \Pi_+\Pi_- = \Pi_-\Pi_+ = 0, \quad \Pi_+ + \Pi_- = 1\,. \tag{4.75}$$

It is also possible to show that for any solution of Dirac equation, the energy projection operators commute with the helicity projection operators,

$$\Big[\Lambda_+(\boldsymbol{p}), \Pi_\pm(\boldsymbol{p})\Big] = \Big[\Lambda_-(\boldsymbol{p}), \Pi_\pm(\boldsymbol{p})\Big] = 0\,. \tag{4.76}$$

These relations are independent of the representation we may choose to use, and this last one shows that we are allowed to choose common eigenstates of $\Lambda_\pm(p)$ and $\Pi_\pm(p)$ as the basis of our representation. For example, if the 3-momentum is in the z-direction, one can easily verify that the spinors given in Eqs. (4.61) and (4.62) are indeed helicity eigenstates:

$$\Sigma_p u_s(p) = s u_s(p)\,, \qquad \Sigma_p v_s(p) = -s v_s(p)\,, \tag{4.77}$$

with $s = \pm$. For any other direction of the momentum, one can make linear combinations of the two u-spinors of Eq. (4.61) and the two v-spinors of Eq. (4.62) such that they satisfy Eq. (4.77). For the u-spinors, therefore, the subscript (+ or −) denotes the helicity. For the v-spinors, it is the opposite of helicity. As we mentioned after Eq. (4.62), the reason for this choice will be explained in §4.6.

□ **Exercise 4.15** Show that, for arbitrary direction of the 3-momentum, the spinors which are simultaneous eigenstates of $\Lambda_\pm$ and $\Pi_\pm$ are given by Eqs. (4.61) and (4.62), where $\chi_\pm$ are given not by Eq. (4.60), but by

$$\chi_+ = \frac{1}{\sqrt{2\mathrm{p}(\mathrm{p}+p_z)}} \begin{pmatrix} \mathrm{p}+p_z \\ p_x + ip_y \end{pmatrix},$$
$$\chi_- = \frac{1}{\sqrt{2\mathrm{p}(\mathrm{p}+p_z)}} \begin{pmatrix} -p_x + ip_y \\ \mathrm{p}+p_z \end{pmatrix}. \tag{4.78}$$

□ **Exercise 4.16** Prove the relations of Eq. (4.75).

□ **Exercise 4.17** ⋆ Prove the relations of Eq. (4.76).

4.4.3 Chirality projection operators

In general, any operator that squares to the identity can be used to construct a projection operator. If Q is an operator such that $Q^2 = 1$, we can define two operators $P_+ = \frac{1}{2}(1+Q)$ and $P_- = \frac{1}{2}(1-Q)$. These operators obey $P_\pm^2 = P_\pm$, $P_+ + P_- = 1$, and $P_+P_- = 0$, i.e., they are projection operators projecting onto orthogonal subspaces. The energy and helicity projection operators $\Lambda_\pm$ and $\Pi_\pm$ are specific examples of this sort of construction. Another obvious operator which squares to the identity is γ_5. We can use γ_5 to construct another pair of projection operators,

$$\mathsf{L} = \frac{1}{2}(1-\gamma_5)\,, \qquad \mathsf{R} = \frac{1}{2}(1+\gamma_5)\,. \tag{4.79}$$

These are called the chirality projection operators. These turn out to be closely related to the helicity projection operators, because in the limit $m \to 0$, the helicity projection operators become chirality projection operators,

$$\Pi_\pm(p) \to \frac{1}{2}\left(1 \pm \gamma^5\right) . \tag{4.80}$$

□ **Exercise 4.18** Show that it is impossible to find a linear combination of the u-spinors of Eq. (4.61) that would be an eigenstate of γ_5, unless the fermion is massless. Further, for a massless fermion, show that the eigenstates of γ_5 are identical to the helicity eigenstates.

□ **Exercise 4.19** $\star$ Prove Eq. (4.80).

4.4.4 Spin projection operators

As mentioned earlier, helicity projection operators project out definite spin projections along the direction of the 3-momentum of a particle. They are therefore useless for a definite spin projection of a particle at rest. For that one needs different projection operators.

Suppose we want to project the spin states for a certain spatial direction denoted by the unit 3-vector $\hat{s}$. For this, first let us define a 4-vector n^μ whose components are given by

$$\begin{aligned} n^0 &= \frac{\boldsymbol{p}\cdot\hat{s}}{m} , \\ \boldsymbol{n} &= \hat{s} + \frac{(\boldsymbol{p}\cdot\hat{s})\boldsymbol{p}}{m(E+m)} , \end{aligned} \tag{4.81}$$

where $(E, \boldsymbol{p})$ denote the 4-momentum of the particle of mass m. Notice that n^μ satisfies the conditions

$$p^\mu n_\mu = 0 , \qquad n^\mu n_\mu = -1 . \tag{4.82}$$

Because of the second property in Eq. (4.82), it is easy to check that the operator $\gamma_5 \not{n}$ squares to unity. So we can construct the projection operators

$$P_\uparrow = \frac{1}{2}(1 + \gamma_5 \not{n}) , \qquad P_\downarrow = \frac{1}{2}(1 - \gamma_5 \not{n}) . \tag{4.83}$$

To understand the significance of these operators, let us consider $\hat{s}$ to be the x-direction, i.e., $\hat{s} = (1, 0, 0)$. In the rest frame of the

particle, $n^\mu = (0, \hat{s})$, so that $\not{n} = \gamma_1$. Using the definition of γ_5 from Eq. (4.12), we get $\gamma_5 \not{n} = i\gamma^0\gamma^2\gamma^3 = \gamma^0\sigma^{23} = \gamma^0\Sigma^1$, where the 3-vector $\mathbf{\Sigma}$ was defined in Eq. (4.72). This commutes with the spin operator in the x-direction, Σ^1, so the eigenstates of this operator are eigenstates of Σ^1 as well. In other words, for this choice of $\hat{s}$, the eigenstates of $\gamma_5 \not{n}$ have well-defined spin components along the x-direction in the rest frame of the particle. The operators given above project out the two possible values of the spin component.

□ **Exercise 4.20** Consider a particle at rest, and take $n^\mu = (0, 1, 0, 0)$ like in the text. Construct $P_\uparrow$ explicitly in the Dirac-Pauli representation of the γ-matrices. Take the u and the v spinors shown in Eq. (4.63). Show that $P_\uparrow u(\mathbf{0})$ and $P_\uparrow v(\mathbf{0})$ give eigenstates of Σ_x with eigenvalues $+1$ and -1 respectively.

□ **Exercise 4.21** Prove the same thing, without taking any explicit form for the Dirac matrices and spinors. [Hint: $u(\mathbf{0})$ and $v(\mathbf{0})$ are eigenstates of γ_0, which follows from Eq. (4.46).]

4.5 Lagrangian for a Dirac field

The problem with a single particle interpretation of the solutions of the Dirac equation is the same as that with a scalar field. It contains negative energy states. So we want to go over to a field theoretic interpretation. The first step for that is to construct a Lagrangian.

The Dirac equation can be derived from a Lagrangian

$$\mathscr{L} = \overline{\psi}(i\not{\partial} - m)\psi \,. \tag{4.84}$$

If we put in the spinor component indices explicitly, this will read

$$\mathscr{L} = \overline{\psi}_\alpha \Big(i(\gamma^\mu)_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta} \Big) \psi_\beta \,. \tag{4.85}$$

From this, one can derive the Euler-Lagrange equations. We are considering the case where ψ is a complex field (see Appendix A.1 for the possibility of a real field). Since components of $\overline{\psi}$ are linear combinations of the components of $\psi^\dagger$, we can treat ψ and $\overline{\psi}$ to be independent. Thus, the Euler-Lagrange equation for $\overline{\psi}$ gives

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \overline{\psi}_\alpha \right)} \right) = \frac{\partial \mathscr{L}}{\partial \overline{\psi}_\alpha} \,. \tag{4.86}$$

The left hand side is zero, since the Lagrangian contains no terms with a derivative of $\overline{\psi}$. Thus we obtain Eq. (4.22).

□ **Exercise 4.22** Derive the Euler-Lagrange equation for ψ and show that it is

$$\overline{\psi}(i\overleftarrow{\partial} + m) = 0\,, \tag{4.87}$$

where the left-arrow sign on the derivative implies that the derivative should operate on things to its left. Then taking the hermitian conjugate of this equation and using the properties of the γ-matrices, show that it is the same equation as that given in Eq. (4.22).

In §2.2 we mentioned that in classical field theories, the action is a real number, a functional of the fields. The Lagrangian is then also a real function in classical field theory. In quantum physics, fields are operators, so the Lagrangian should be hermitian. Indeed, the Lagrangians for scalar fields, given in Eqs. (3.5) and (3.45), are hermitian. But the Dirac Lagrangian of Eq. (4.84) is not. The mass term is hermitian, but the hermitian conjugate of the term involving derivatives is not equal to itself:

$$\begin{aligned}\left(i\overline{\psi}\gamma^\mu\partial_\mu\psi\right)^\dagger &= -i(\partial_\mu\psi)^\dagger\gamma^{\mu\dagger}\gamma_0\psi \\ &= -i(\partial_\mu\overline{\psi})\gamma^\mu\psi\,,\end{aligned} \tag{4.88}$$

using the hermiticity property of the Dirac matrices, Eq. (4.18), in the last step. If we really want a hermitian Lagrangian, we can discard the Lagrangian $\mathscr{L}$ of Eq. (4.84) and use instead

$$\mathscr{L}' = \frac{i}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}(\partial_\mu\overline{\psi})\gamma^\mu\psi - m\overline{\psi}\psi\,. \tag{4.89}$$

However, it is not absolutely essential that we use this hermitian form of the Lagrangian. The reason is that

$$\mathscr{L} - \mathscr{L}' = \frac{i}{2}\overline{\psi}\gamma^\mu\partial_\mu\psi + \frac{i}{2}(\partial_\mu\overline{\psi})\gamma^\mu\psi = \partial_\mu\left(\frac{i}{2}\overline{\psi}\gamma^\mu\psi\right)\,, \tag{4.90}$$

i.e., the difference is a total divergence. According to the discussion of §2.2, the two Lagrangians should therefore be equivalent. We will therefore keep using Eq. (4.84) for the Lagrangian of a Dirac field.

The Dirac Lagrangian is invariant under the transformations

$$\psi \to e^{-iq\theta}\psi\,, \tag{4.91}$$

for which the Noether current is

$$j^\mu = q\overline{\psi}\gamma^\mu\psi\,. \tag{4.92}$$

The conserved charge is given by

$$Q = q \int d^3x \, \overline{\psi}\gamma^0\psi = q \int d^3x \, \psi^\dagger\psi \,. \tag{4.93}$$

□ **Exercise 4.23** Find the equations of motion and the conserved current for the Lagrangian of Eq. (4.89).

□ **Exercise 4.24** ⋆ The Pauli-Lubansky vector is defined by

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}P^\nu J^{\lambda\rho} \,, \tag{4.94}$$

where P^ν is the momentum operator. A theorem then states that, for particles which are eigenstates of momentum and angular momentum,

$$W^\mu W_\mu = -m^2 s(s+1) \,, \tag{4.95}$$

where s is called the spin of the particle. Using the expression for $J_{\mu\nu}$ from Eq. (4.39), show that the spin of a particle satisfying the Dirac equation is $\frac{1}{2}$. [Hint: The contraction formulas of the antisymmetric tensor are given in the Appendix.]

4.6 Fourier decomposition of the field

We now Fourier decompose the Dirac field as we did with the scalar field in the last chapter. As with the scalar field, we write the Dirac field as an integral over momentum space of the plane wave solutions, with creation and annihilation operators as coefficients,

$$\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=1,2} \left(f_s(\boldsymbol{p})u_s(\boldsymbol{p})e^{-ip\cdot x} + \widehat{f}_s^\dagger(\boldsymbol{p})v_s(\boldsymbol{p})e^{ip\cdot x} \right) . \tag{4.96}$$

Of course, this also implies

$$\overline{\psi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=1,2} \left(f_s^\dagger(\boldsymbol{p})\overline{u}_s(\boldsymbol{p})e^{ip\cdot x} + \widehat{f}_s(\boldsymbol{p})\overline{v}_s(\boldsymbol{p})e^{-ip\cdot x} \right) . \tag{4.97}$$

The total Hamiltonian can be written as

$$\begin{aligned} H &= \int d^3x \, \frac{\delta L}{\delta\dot{\psi}_\alpha(x)}\dot{\psi}_\alpha(x) - L \\ &= \int d^3x \, \overline{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi \,, \end{aligned} \tag{4.98}$$

using the Lagrangian of Eq. (4.84). To apply the operator $(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)$ on $\psi(x)$ given in Eq. (4.96), we note that

$$\begin{aligned}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)u_s(\boldsymbol{p})e^{-ip\cdot x} &= (\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)u_s(\boldsymbol{p})e^{-ip\cdot x} \\ &= \gamma_0 E_p u_s(\boldsymbol{p})e^{-ip\cdot x}\,, \end{aligned} \tag{4.99}$$

using the definition of the u-spinors from Eq. (4.46) at the last step. Similarly,

$$(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)v_s(\boldsymbol{p})e^{ip\cdot x} = -\gamma_0 E_p v_s(\boldsymbol{p})e^{ip\cdot x}\,. \tag{4.100}$$

Substituting these in Eq. (4.98) and using $\overline{u}\gamma_0 = u^\dagger$ and $\overline{v}\gamma_0 = v^\dagger$, we obtain

$$\begin{aligned}H = \int \frac{d^3x}{2(2\pi)^3}\int d^3p'\int d^3p\,\sqrt{\frac{E_p}{E_{p'}}}\,\times \\ \sum_{s,s'=1,2}\left(f^\dagger_{s'}(\boldsymbol{p}')u^\dagger_{s'}(\boldsymbol{p}')e^{ip'\cdot x}+\widehat{f}_{s'}(\boldsymbol{p}')v^\dagger_{s'}(\boldsymbol{p}')e^{-ip'\cdot x}\right) \\ \left(f_s(\boldsymbol{p})u_s(\boldsymbol{p})e^{-ip\cdot x}-\widehat{f}^\dagger_s(\boldsymbol{p})v_s(\boldsymbol{p})e^{ip\cdot x}\right)\,. \end{aligned} \tag{4.101}$$

Notice that the cross terms, i.e., the ones involving $u^\dagger v$ and $v^\dagger u$, vanish due to the normalization conditions from Eq. (4.49). The other two terms give

$$H = \int d^3p\; E_p \sum_{s=1,2}\left[f^\dagger_s(\boldsymbol{p})f_s(\boldsymbol{p})-\widehat{f}_s(\boldsymbol{p})\widehat{f}^\dagger_s(\boldsymbol{p})\right]\,. \tag{4.102}$$

This Hamiltonian can give negative energy eigenvalues, which is a serious problem. And this does not go away by normal-ordering, at least if we assume the definition of normal-ordering employed in Ch. 3. The way to take care of this problem is to assume that the creation and annihilation operators in this case obey *anticommutation* rather than commutation relations:

$$\left[f_s(\boldsymbol{p}),f^\dagger_{s'}(\boldsymbol{p}')\right]_+ = \left[\widehat{f}_s(\boldsymbol{p}),\widehat{f}^\dagger_{s'}(\boldsymbol{p}')\right]_+ = \delta_{ss'}\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \tag{4.103}$$

while all other *anticommutators* are zero. And we define normal ordering for fermionic operators by saying that whenever we encounter products of them, we shall put the annihilation operators to the right and creation operators to the left as if the *anticommutators* were zero.

Using this prescription, then, we obtain the normal-ordered Hamiltonian to be

$$:H: = \int d^3p\, E_p \sum_{s=1,2} \left[f_s^\dagger(\boldsymbol{p}) f_s(\boldsymbol{p}) + \widehat{f}_s^\dagger(\boldsymbol{p}) \widehat{f}_s(\boldsymbol{p}) \right] . \qquad (4.104)$$

□ **Exercise 4.25** Using the anticommutation relation of Eq. (4.103), show that the field operators satisfy the relation

$$\left[\psi(t, \boldsymbol{x}), \psi^\dagger(t, \boldsymbol{y}) \right]_+ = \delta^3(\boldsymbol{x} - \boldsymbol{y}) , \qquad (4.105)$$

whereas all other anticommutators vanish.

□ **Exercise 4.26** ⋆ Derive the expression for $\mathscr{M}^\lambda_{\mu\nu}$ for a Dirac field. Show that the conserved charges are the same as those given in Eq. (4.39), i.e.,

$$[\psi, J_{\mu\nu}]_- = \left(i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right) \psi . \qquad (4.106)$$

□ **Exercise 4.27** There are different conventions regarding the normalization for the spinors. One can use a quantity N_1 in place of $2E_p$ on the right hand side of the normalization equations, Eq. (4.49). Moreover, one can define the anticommutation relations of Eq. (4.103) with an extra factor of N_2 on the right hand side, and use a factor N_3 instead of $2E_p$ under the square root sign in the denominator of the Fourier expansion in Eq. (4.96). Show that these quantities must be related by

$$N_1 N_2 = N_3 , \qquad (4.107)$$

in order that Eq. (4.105) is satisfied.

Some consequences of the anticommutation relations and the new definition of normal-ordering are worth exploring at this point. First, notice that the Noether charge, in the normal-ordered form, becomes

$$\begin{aligned} :Q: &= q \int d^3x\ :\psi^\dagger \psi: \\ &= q \int d^3p \sum_{s=1,2} \left[f_s^\dagger(\boldsymbol{p}) f_s(\boldsymbol{p}) - \widehat{f}_s^\dagger(\boldsymbol{p}) \widehat{f}_s(\boldsymbol{p}) \right] . \end{aligned} \qquad (4.108)$$

Just as in the case of complex scalar fields, we see that the Noether charge for the particles created by $f^\dagger$ is opposite to the charge of those created by $\widehat{f}^\dagger$. The latter are therefore antiparticles of the former ones.

Secondly, we said that all anticommutators other than the ones mentioned in Eq. (4.103) are zero. For example, if we take the anticommutator of $f_s^\dagger(\boldsymbol{p})$ with itself, it will vanish. So will the anticommutator of $\widehat{f}_s^\dagger(\boldsymbol{p})$ with itself. This means that

$$f_s^\dagger(\boldsymbol{p}) f_s^\dagger(\boldsymbol{p}) = \widehat{f}_s^\dagger(\boldsymbol{p}) \widehat{f}_s^\dagger(\boldsymbol{p}) = 0\,. \tag{4.109}$$

In other words, we cannot create two particles or two antiparticles with the same spin and same momentum. Anticommutators imply the *Pauli exclusion principle.*

We can also define the Fock space for fermions by starting with the definition of the vacuum as

$$f_s(\boldsymbol{p})\,|0\rangle = 0\,, \quad \widehat{f}_s(\boldsymbol{p})\,|0\rangle = 0 \qquad \text{for all } \boldsymbol{p} \text{ and } s. \tag{4.110}$$

The states containing particles and antiparticles can be constructed by acting on the vacuum by $f^\dagger$ and $\widehat{f}^\dagger$. The Hamiltonian of Eq. (4.104) and the anticommutation relations of Eq. (4.103) produce the following relations:

$$\begin{aligned} [H, f_r^\dagger(\boldsymbol{k})]_- &= E_k f^\dagger(\boldsymbol{k})\,, \\ [H, \widehat{f}_r^\dagger(\boldsymbol{k})]_- &= E_k \widehat{f}^\dagger(\boldsymbol{k})\,. \end{aligned} \tag{4.111}$$

These imply that both $f_r^\dagger(\boldsymbol{k})$ and $f_r^\dagger(\boldsymbol{k})$ create positive energy states, a particle in the first case and an antiparticle in the second. In particular, $f_r^\dagger(\boldsymbol{k})\,|0\rangle$ is a one-particle state, and $\widehat{f}_r^\dagger(\boldsymbol{k})\,|0\rangle$ is a one-antiparticle state.

To understand the helicities of these one-particle states, we first note that one can use the normalization conditions in Eq. (4.49) to invert the plane-wave expansion for the field operator $\psi(x)$, given in Eq. (4.96), and write

$$\begin{aligned} f_r(\boldsymbol{k}) &= \frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x\, e^{-ik\cdot x} u_r^\dagger(\boldsymbol{k}) \psi(x)\,, \\ \widehat{f}_r^\dagger(\boldsymbol{k}) &= \frac{1}{\sqrt{(2\pi)^3 2E_k}} \int d^3x\, e^{-ik\cdot x} v_r^\dagger(\boldsymbol{k}) \psi(x)\,. \end{aligned} \tag{4.112}$$

Secondly, if we define the operator

$$J_{\boldsymbol{k}} = \frac{\boldsymbol{J}\cdot\boldsymbol{k}}{\mathbf{k}}\,, \tag{4.113}$$

its commutation with the field operator can be easily derived from Eq. (4.106) to be

$$[\psi(x), \boldsymbol{J}_{\boldsymbol{k}}]_- = \frac{1}{2}\Sigma_{\boldsymbol{k}}\psi(x)\,, \tag{4.114}$$

since the orbital angular momentum part in $J_{\mu\nu}$ does not contribute to $\boldsymbol{J}_{\boldsymbol{k}}$. Here $\Sigma_{\boldsymbol{k}}$ is the helicity operator, defined in Eq. (4.73). The corresponding commutators with $f_r(\boldsymbol{k})$ and $\widehat{f}_r^\dagger(\boldsymbol{k})$ can then be obtained by using the two relations in Eq. (4.112). For example, the second of these relations gives

$$\left[\widehat{f}_r^\dagger(\boldsymbol{k}), \boldsymbol{J}_{\boldsymbol{k}}\right]_- = \frac{1}{2}\,\frac{1}{\sqrt{(2\pi)^3 2E_k}}\int d^3x\, e^{-ik\cdot x} v_r^\dagger(\boldsymbol{k})\Sigma_{\boldsymbol{k}}\psi(x)\,. \tag{4.115}$$

If we now use the convention of the helicity eigenstates as in Eq. (4.77) where $r = \pm$, we find

$$\left[\boldsymbol{J}_{\boldsymbol{k}}, \widehat{f}_r^\dagger(\boldsymbol{k})\right]_- = \frac{1}{2} r \widehat{f}_r^\dagger(\boldsymbol{k})\,. \tag{4.116}$$

Operating on the vacuum which has no angular momentum, we find that the state $\widehat{f}_r^\dagger(\boldsymbol{k})\,|0\rangle$ has a spin-projection $\frac{1}{2}r$ in the direction of the momentum, which implies a helicity r. This, in hindsight, is the rationale for the convention of choosing the subscripts on the v-spinors to be opposite to their helicity: v_+, for example, has negative helicity, but the subscript is governed by the fact that the creation operator that accompanies it in the Fourier expansion of Eq. (4.96) in fact creates a positive-helicity antiparticle state from the vacuum. The helicities of particle states can be calculated in the same way.

□ **Exercise 4.28** Show that $f_r^\dagger(\boldsymbol{k})$ creates a particle state of helicity r.

4.7 Propagator

Now we are interested in how the fermionic states move in space-time. We proceed as we did for the scalar field and write the Dirac equation coupled to a source:

$$(i\gamma^\mu\partial_\mu - m)\,\psi(x) = J(x)\,, \tag{4.117}$$

and introduce the Dirac propagator by

$$\left(i\gamma^\mu\partial_\mu^x - m\right) S(x - x') = \delta^4(x - x')\,. \tag{4.118}$$

Then the solution of the Dirac equation is given by

$$\psi(x) = \psi_0(x) + \int d^4x'\, S(x-x')J(x')\,, \tag{4.119}$$

where $\psi_0(x)$ is any solution of the homogeneous Dirac equation, Eq. (4.22).

Our technique for solving Eq. (4.118) is the same as the one used for scalar fields. We take a Fourier transform to write

$$S(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')} S(p)\,. \tag{4.120}$$

Then the Fourier transform, $S(p)$, satisfies the equation

$$(\not{p} - m)\, S(p) = 1\,. \tag{4.121}$$

Multiplying both sides by $\not{p} + m$, we obtain

$$\left(p^2 - m^2\right) S(p) = \not{p} + m\,. \tag{4.122}$$

This is a matrix equation, $S(p)$ is a matrix. But $p^2 - m^2$ is a number, so we can take it to the denominator on the right hand side. We immediately see that there are poles for $p^0 = \pm E_p$, which will cause problems for the Fourier transform. So we use the ϵ prescription as for the scalar field to get the so-called Feynman propagator:

$$S_F(p) = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}\,. \tag{4.123}$$

Often, for the sake of brevity, this is written as

$$S_F(p) = \frac{1}{\not{p} - m + i\varepsilon}\,, \tag{4.124}$$

directly from Eq. (4.121), which should be taken to mean the same thing as Eq. (4.123).

If we put the Fourier transform obtained in Eq. (4.123) into Eq. (4.120), we get

$$\begin{aligned} S_F(x-x') &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')} \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} \\ &= (i\not{\partial} + m)\Delta_F(x-x')\,, \end{aligned} \tag{4.125}$$

where $\Delta_F(x-x')$ is the propagator for the scalar fields derived in Eq. (3.76). This gives

$$iS_F(x-x') = \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\Theta(t-t')(\not{p}+m)e^{-ip\cdot(x-x')} - \Theta(t'-t)(\not{p}-m)e^{ip\cdot(x-x')}\right]. \quad (4.126)$$

The matrix elements of $S_F(x-x')$ can be written in terms of the field operators in the form

$$iS_{F\alpha\beta}(x-x') = \left\langle 0 \left| \mathscr{T}\left[\psi_\alpha(x)\overline{\psi}_\beta(x')\right] \right| 0 \right\rangle , \quad (4.127)$$

where the time-ordered product for fermionic fields are defined by

$$\mathscr{T}\left[\psi_\alpha(x)\overline{\psi}_\beta(x')\right] \equiv \begin{cases} \psi_\alpha(x)\overline{\psi}_\beta(x') & \text{if } t > t' \\ -\overline{\psi}_\beta(x')\psi_\alpha(x) & \text{if } t' > t\,. \end{cases} \quad (4.128)$$

Like the propagator for the complex scalar field, the propagator in Eq. (4.127) describes either a particle going from $(t', \boldsymbol{x}')$ to $(t, \boldsymbol{x})$, or an antiparticle going from $(t, \boldsymbol{x})$ to $(t', \boldsymbol{x}')$.

□ **Exercise 4.29** Derive Eq. (4.126) from Eq. (4.125).

□ **Exercise 4.30** Prove Eq. (4.127).

Chapter 5

The S-matrix expansion

So far we have dealt with *free* fields only. Even though we had a source term for the derivation of the propagator, the source was considered non-dynamical for that purpose, i.e., the source itself did not have a propagator. A truly free field has no experimental consequences, since it would not interact with the detection apparatus at all. Moreover, if all fields were free fields, they would exist independent of one another and nothing would happen in the world!

Of course, the fields in the real world are not free. They interact with one another, which gives rise to phenomena we can observe. To describe such phenomena, we therefore need a framework to describe the interactions.

In the Lagrangians of free fields described earlier, we had only terms which are quadratic in field operators. In terms of creation and annihilation operators, this is exactly what is needed for describing free fields. At any space-time point, the free Hamiltonian can annihilate a particle and create the same. Thus, in effect, the same particle travels on.

It is possible to have terms bilinear in two fields which would annihilate a particle of one type and create a particle of another type. These can be brought around to the standard quadratic form by a redefinition of fields. Alternatively, they can be treated as quadratic interactions of the original fields.

To describe any other type of interaction, we need some term in the Lagrangian which has three or more field operators. Moreover, the term has to be Poincaré invariant since the Lagrangian (density) is. Also, the action is a real number, so the Lagrangian has to be

hermitian. Of course, as in the case of the Dirac field, the Lagrangian can differ from a hermitian Lagrangian by total divergence terms.

5.1 Examples of interactions

The possibilities of interaction depend, of course, on the type of fields in a theory. For example, suppose we have only one real scalar field ϕ in our theory. Then in addition to the free Lagrangian we can have the following interaction terms:

$$\mathscr{L}_{\text{int}} = -\mu\phi^3 - \lambda\phi^4 - \sum_{n\geq 5} \lambda_{(n)}\phi^n \,. \tag{5.1}$$

Here μ, λ etc are some constants which are called *coupling constants*, i.e., their values determine how strong the relevant interactions are. We have lumped all the terms with more than four powers of ϕ together under the summation sign. This is because the coupling constants in these terms have negative mass dimensions. Such couplings introduce some problems in field theory, which we will discuss later.

□ **Exercise 5.1** Show that the mass dimension of the coupling constant $\lambda_{(n)}$ is $4-n$.

If we have a complex scalar field, the interaction terms in Eq. (5.1) can still be present. In addition, we can also have terms where one or more factors of ϕ is replaced with $\phi^\dagger$. Of course, the Lagrangian must remain hermitian after these replacements. The free Lagrangian of a complex scalar field is invariant under the change of the phase of the field, which was mentioned in Eq. (3.54). If for some reason we want to have this symmetry for the interaction terms as well, then the most general form of the interactions would be

$$\mathscr{L}_{\text{int}} = -\sum_{n\geq 2} \lambda_{(2n)} \left(\phi^\dagger\phi\right)^n \tag{5.2}$$

If we have only spinor fields in a theory, our choices are much more restricted. This is because the spinors transform non-trivially under Lorentz transformations, and we must make combinations which are Lorentz invariant. A product of an odd number of spinor fields cannot be Lorentz invariant. Thus the first non-trivial combination will

have four fermion fields. We can also have more than four if we so wish.

There can be many types of 4-fermion interactions. For example, we can write an interaction of the form

$$\mathscr{L}_{\rm int} = G\,\overline{\psi}\psi\,\overline{\psi}\psi\,. \tag{5.3}$$

However, there are other possibilities. For example, we can also write

$$\mathscr{L}_{\rm int} = G\,\overline{\psi}\gamma^{\mu}\psi\,\overline{\psi}\gamma_{\mu}\psi \tag{5.4}$$

or

$$\mathscr{L}_{\rm int} = G\,\overline{\psi}\sigma^{\mu\nu}\psi\,\overline{\psi}\sigma_{\mu\nu}\psi\,. \tag{5.5}$$

We can also make combinations using γ_5, e.g.:

$$\mathscr{L}_{\rm int} = G\,\overline{\psi}\gamma^{\mu}\psi\,\overline{\psi}\gamma_{\mu}\gamma_5\psi \tag{5.6}$$

and so on. One combination which played a crucial role in the development of weak interaction theory is

$$\mathscr{L}_{\rm int} = G\,\overline{\psi}\gamma^{\mu}(1-\gamma_5)\psi\,\overline{\psi}\gamma_{\mu}(1-\gamma_5)\psi\,. \tag{5.7}$$

The fields denoted by ψ in these equations need not refer to the same fermion field. For example, suppose we want to describe nuclear beta decay in which a neutron decays into a proton, an electron and an electron-antineutrino. In this case, the last equation turns out to be successful in explaining experimental data provided we replace the four ψ-operators as follows:

$$\mathscr{L}_{\rm int} = \frac{G_\beta}{\sqrt{2}}\,\overline{\psi}_{(e)}\gamma^{\mu}(1-\gamma_5)\psi_{(\nu_e)}\,\overline{\psi}_{(p)}\gamma_{\mu}(1-\gamma_5)\psi_{(n)} + \text{h.c.}\,, \tag{5.8}$$

where the subscripts in parentheses stand for the particles involved. The factor of $\sqrt{2}$ in the coupling constant is purely conventional. Defined this way, G_β is called the β-decay constant. Notice also that in Eq. (5.8), we have added the symbol h.c.. This is because the term written down in detail is not hermitian by itself. Since the Lagrangian has to be hermitian, this means that the hermitian conjugate term must also be present in the Lagrangian.

Let us now consider a field theory in which there is one real scalar field ϕ and a spinor field ψ. In this case, apart from the interactions

that involve only the ϕ-terms or only the ψ-terms, we can have other interaction terms which involve both ϕ and ψ. For example, we can have

$$\mathscr{L}_{\text{int}} = -h\bar{\psi}\psi\phi\,. \tag{5.9}$$

In the standard model of particle interactions, this kind of interaction arises between the fermions and a scalar boson field representing the *Higgs particle*. Similarly, one can have

$$\mathscr{L}_{\text{int}} = -h'\bar{\psi}\gamma_5\psi\phi\,. \tag{5.10}$$

Originally, a term like this was proposed to describe the interaction between protons and neutrons (which are together grouped as *nucleons* since they occur in atomic nuclei) with spinless particles called *pions*.

□ **Exercise 5.2** Show that, for the interaction Lagrangian to be hermitian, the coupling constant h in Eq. (5.9) has to be real, and h' in Eq. (5.10) has to be purely imaginary.

In writing all these interactions, we have ignored the free terms in the Lagrangian since they are already known to us. It is the nature of the interaction that defines a specific theory, and so theories are named after their interaction terms. The type of interaction given in Eq. (5.10) was first envisaged by Yukawa. In the modern literature, interactions of this form and also of the form in Eq. (5.9) are together called Yukawa interactions. Any theory with a scalar and a fermion field with such interactions is called Yukawa theory. The interaction with four fermionic field operators was first discussed by Fermi and are therefore called Fermi interactions, or sometimes 4-fermion interactions. With the scalar field, we can have a $\lambda\phi^4$ theory where the only interaction is the ϕ^4 interaction. Discussion of these and other interactions requires a general framework to deal with interacting fields, which we now proceed to construct.

5.2 Evolution operator

If for a given system we could find all the quantum states we would be able to exactly predict the behavior of the system from its initial conditions. In other words, given an initial state $|i)$ consisting of

some particles with given quantum numbers we would know all the possible states $|f\rangle$ it could evolve into, along with the amplitudes of transition. Unfortunately, apart from a few and almost always unrealistic exceptions, it is not possible to find the eigenstates of a Hamiltonian if the fields in it interact with each other. Therefore, given a theory, we separate the Hamiltonian into a part which we can solve exactly and another which we cannot,

$$H = H_0 + H_I \,. \tag{5.11}$$

Of course we could have done the separation in the Hamiltonian density as well. Here H_0 includes all the bits of H whose eigenstates are exactly known to us. We shall restrict ourselves where H_0 is the free Hamiltonian of the fields concerned, which can be written in terms of the creation and annihilation operators as described earlier. The remaining piece, H_I, contains all the interactions. We want to find some way of describing the interactions in terms of the free fields. We shall have to resort to perturbation expansions in order to achieve this. There can be situations where H_0 includes the Hamiltonian for bound states, describing an atom for example, and H_I describes interactions of particles with the atom. We shall not consider these cases, but the formalism we shall develop can be applied equally well to them.

A state evolving under the total Hamiltonian $H = H_0 + H_I$ obeys

$$i\frac{d}{dt}\,|\Psi(t)\rangle = (H_0 + H_I)\,|\Psi(t)\rangle \,. \tag{5.12}$$

On the other hand in the absence of interactions, the evolution of states is governed by the free Hamiltonian H_0, which is constructed out of free field operators. The states then obey the equation

$$i\frac{d}{dt}\,|\Psi_0(t)\rangle = H_0\,|\Psi_0(t)\rangle \,. \tag{5.13}$$

Let us now define two different evolution operators $U_0(t)$ and $U(t)$ such that

$$\begin{aligned} |\Psi_0(t)\rangle &= U_0(t)\,|\Psi_0(-\infty)\rangle \equiv U_0(t)\,|i\rangle \,, \\ |\Psi(t)\rangle &= U_0(t)U(t)U_0^\dagger(t)\,|\Psi_0(t)\rangle \,. \end{aligned} \tag{5.14}$$

If the states on both sides are taken to be normalized states, the operators $U_0(t)$ and $U(t)$ are unitary for any t. The initial state $|i\rangle$

is a member of some Fock space. Then

$$|\Psi(t)\rangle = U_0(t)U(t)\,|i\rangle\ . \tag{5.15}$$

We can write Eq. (5.13), which is Schrödinger's equation for the free system, as

$$i\frac{d}{dt}U_0(t)\,|i\rangle = H_0U_0(t)\,|i\rangle\ . \tag{5.16}$$

By considering a complete set of states at $t = -\infty$, we can convert this into an operator equation,

$$i\frac{d}{dt}U_0(t) = H_0U_0(t)\,, \tag{5.17}$$

which implies

$$U_0(t) = e^{-iH_0t}\,, \tag{5.18}$$

assuming H_0 is not explicitly dependent on time.

Let us now consider a system evolving under the full Hamiltonian. Schrödinger's equation for this system is

$$i\frac{d}{dt}\,|\Psi(t)\rangle = (H_0 + H_I)\,|\Psi(t)\rangle\ . \tag{5.19}$$

We can simplify the left hand side by using Eq. (5.15),

$$\begin{aligned} i\frac{d}{dt}\,|\Psi(t)\rangle &= i\frac{d}{dt}\Big(U_0(t)U(t)\Big)\,|i\rangle \\ &= \left(i\frac{dU_0(t)}{dt}U(t) + U_0(t)i\frac{dU(t)}{dt}\right)|i\rangle \\ &= \left(H_0U_0(t)U(t) + U_0(t)i\frac{dU(t)}{dt}\right)|i\rangle\ , \end{aligned} \tag{5.20}$$

while the right hand side reads

$$(H_0 + H_I)\,|\Psi(t)\rangle = (H_0 + H_I)\,U_0(t)U(t)\,|i\rangle\ . \tag{5.21}$$

So we can now equate the two sides, and going to a complete set of initial states (at $t = -\infty$) as before, we can write Eq. (5.19) as an operator equation,

$$H_0U_0(t)U(t) + U_0(t)i\frac{dU(t)}{dt} = H_0U_0(t)U(t) + H_IU_0(t)U(t). \tag{5.22}$$

The first term on the left cancels the first term on the right, and multiplying both sides by $U_0^\dagger(t)$, we get

$$\begin{aligned} i\frac{dU(t)}{dt} &= U_0^\dagger(t) H_I U_0(t) U(t) \\ &= H_I(t) U(t) \,, \end{aligned} \tag{5.23}$$

where

$$H_I(t) \equiv U_0^\dagger(t) H_I U_0(t) \,. \tag{5.24}$$

Let us try to understand the meaning of $H_I(t)$. The Hilbert space on which both sides of Eq. (5.23) act is the set of states which we had denoted by $|i\rangle$ earlier. Let us denote the basis vectors in this space by $|\beta\rangle$. These are then the time-independent free states of definite quantum numbers. If we now look at the matrix elements of $H_I(t)$ between these basis states, we find that

$$\begin{aligned} \langle \beta \left| H_I(t) \right| \beta' \rangle &= \left\langle \beta \left| U_0^\dagger(t) H_I U_0(t) \right| \beta' \right\rangle \\ &= \langle \beta, t \left| H_I \right| \beta', t \rangle \,. \end{aligned} \tag{5.25}$$

Here we have written

$$U_0(t) \left| \beta \right\rangle = \left| \beta, t \right\rangle \,, \tag{5.26}$$

the time-dependent free state into which $|\beta\rangle$ would have evolved at time t had there been no interactions. So the matrix elements of $H_I(t)$ between time-independent basis states are equal to those of H_I between the corresponding time-dependent free states. In other words, $H_I(t)$ is the interaction Hamiltonian H_I, now written in terms of free fields at time t. Sometimes this is referred to as the 'interaction Hamiltonian in the interaction picture'. Eq. (5.23) is solved by an integral equation,

$$U(t) = U(t_0) - i \int_{t_0}^{t} dt_1 \; H_I(t_1) U(t_1) \,. \tag{5.27}$$

Now we make an assumption, that the states $|\beta\rangle$ are also a complete set of eigenstates for the full Hamiltonian H as $t \to -\infty$. In other words, $H_I \to 0$ as $t \to -\infty$. We could think of this as saying that all fields are free fields in the remote past, and the interactions

are slowly turned on. This makes sense in scattering experiments, where we have particles moving in as free particles, interacting briefly, and moving away as free particles again. Our assumption boils down to $U(t) \to 1$ as $t \to -\infty$.

Putting $t_0 = -\infty$ in Eq. (5.27), we can rewrite it as

$$U(t) = 1 - i \int_{-\infty}^{t} dt_1 \; H_I(t_1) U(t_1) \,. \tag{5.28}$$

This is an exact equation obeyed by $U(t)$, but is useless in this form because of the presence of $U(t_1)$ on the right hand side. We can obtain a solution iteratively if we replace the $U(t_1)$ on the right hand side by Eq. (5.28). This gives

$$U(t) = 1 - i \int_{-\infty}^{t} dt_1 \; H_I(t_1) \left(1 - i \int_{-\infty}^{t_1} dt_2 \; H_I(t_2) U(t_2) \right) . \tag{5.29}$$

Replacing $U(t_2)$ again by Eq. (5.28) and carrying on this procedure indefinitely, we obtain

$$U(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \; H_I(t_1) H_I(t_2) \cdots H_I(t_n) \,. \tag{5.30}$$

This can be written in a much nicer form if we use the time-ordered products of operators. For example, take the term in the sum with $n = 2$. Note that

$$\begin{aligned} \frac{1}{2!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \; \mathscr{T} \left[H_I(t_1) H_I(t_2) \right] \\ = \frac{1}{2!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \; H_I(t_1) H_I(t_2) \\ + \frac{1}{2!} \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \; H_I(t_2) H_I(t_1) \,. \end{aligned} \tag{5.31}$$

The first term is obtained for $t_1 > t_2$, and the second for $t_2 > t_1$. Now, by interchanging the dummy variables t_1 and t_2 in the second integral and adding up the two integrals, this reduces to

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \; H_I(t_1) H_I(t_2) \,. \tag{5.32}$$

Similarly, the higher terms can also be rewritten in terms of time-ordered products, and the perturbative solution for $U(t)$ can be written as

$$U(t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \cdots \int_{-\infty}^{t} dt_n \, \mathscr{T}\left[H_I(t_1)H_I(t_2)\cdots H_I(t_n)\right]. \quad (5.33)$$

Time ordering of n operators has the obvious meaning that any operator appears to the right of all other operators at later times. One sometimes writes $U(t)$ in the more compact form

$$U(t) = \mathscr{T}\left[\exp\left(-i\int_{-\infty}^{t} dt'\, H_I(t')\right)\right], \quad (5.34)$$

which is to be interpreted to mean Eq. (5.33).

The expression for $U(t)$, Eq. (5.33), would be exact if the series on the right hand side converged. In general, for arbitrary H_I, the series does not converge. As a result, we are usually interested in cases where H_I is proportional to some 'small' parameter. (Here, as in quantum mechanics, 'small' implies that the expectation value of H_I in any state is small compared to the typical energy scale of the system.) However, it turns out that even then, for most theories the expression for $U(t)$ does not converge. Instead of a convergent series, we get an *asymptotic* series, one in which each term is smaller than the preceding one, but the sum does not necessarily converge. One could therefore question the validity of the formalism derived above, as well as physical results which we shall compute by using it. This is not a trivial issue, and as a matter of fact this question survives all known ways of quantizing field theories, not just the canonical quantization that we have done. The response to this is that the test of any formalism is in its agreement with experiments. It is truly remarkable that quantum field theory, despite some severe mathematical problems in defining it, passes the test of experimental verification with flying colors.

5.3 S-matrix

The S-matrix is defined as the limit

$$S = \lim_{t\to\infty} U(t) = \mathscr{T}\left[\exp\left(-i\int_{-\infty}^{\infty} dt\, H_I(t)\right)\right]. \quad (5.35)$$

Since

$$H_I(t) = \int d^3x \, \mathscr{H}_I(x) \tag{5.36}$$

where $x^0 = t$ on the right hand side, we can also write the definition of the S-matrix in a neater form:

$$S = \mathscr{T}\left[\exp\left(-i\int d^4x \, \mathscr{H}_I(x)\right)\right]. \tag{5.37}$$

We have already shown in Eq. (5.14) that $U(t)$ is unitary for any t. Thus the S-matrix is unitary:

$$SS^\dagger = S^\dagger S = 1\,. \tag{5.38}$$

Why is the S-matrix useful? Particle interactions usually occur for small intervals of time. Long before this time, one can consider the particles to be essentially free. Similarly, long after the interaction period, the particles are again essentially free. A typical problem of interaction can then be reduced to finding out whether a given state of free particles, through interactions, can evolve to another given state of free particles long after all interactions have ceased.

Consider a state at $t = -\infty$. Let us denote this initial state by $|i\rangle$. This is a state containing non-interacting fields, with $H_I = 0$. Now let us slowly turn on the interaction, let the state evolve under the full Hamiltonian H, and again slowly turn off the interaction. After a long time the system must again be free and we can describe it by a superposition of states of the form $U_0(t)\,|\beta\rangle$ where $|\beta\rangle$ are our time-independent basis states of free particles (free particle eigenstates of H_0 with definite quantum numbers). Using Eq. (5.15), we find that the amplitude for transition from $|i\rangle$ to another specific state $|f\rangle$ ($|i\rangle, |f\rangle \in \{|\beta\rangle\}$) at late times $t \to \infty$ is

$$\lim_{t\to\infty}\left\langle f\left|U_0^\dagger(t)U_0(t)U(t)\right|i\right\rangle = \lim_{t\to\infty}\langle f\,|U(t)|\,i\rangle = \langle f\,|S|\,i\rangle\,, \tag{5.39}$$

which is just a matrix element of the operator S in the vector space of free states.

We have been a bit sloppy over the business of slowly turning the interaction on or off. It is physically understandable as well as justified from an experimentalist's point of view, but we had already assumed in our derivation that the Hamiltonian did not depend explicitly on time, so we should not really include a time dependent

part. An alternative derivation involves defining 'in' and 'out' states, $|\Psi_{\rm in}(\alpha)\rangle$ and $|\Psi_{\rm out}(\alpha)\rangle$ which are *time-independent* eigenstates of the full Hamiltonian H with quantum numbers collectively denoted by α, but consisting of particles which are too far apart to interact, at $t \to -\infty$ and $t \to \infty$, respectively. The 'in' and 'out' states are separately taken to be orthonormal and complete. The S-matrix is defined as the infinite matrix with elements

$$S_{\beta\alpha} = \langle \Psi_{\rm out}(\beta) | \Psi_{\rm in}(\alpha) \rangle \ . \tag{5.40}$$

While a derivation of the S-matrix using 'in' and 'out' states is more rigorous, it turns out that our free states, the eigenstates $|\beta\rangle$ of the free Hamiltonian H_0, are related to the 'in' and 'out' states by

$$|\Psi_{\rm in}(\beta)\rangle = \Omega(-\infty)\,|\beta\rangle\,, \qquad |\Psi_{\rm out}(\beta)\rangle = \Omega(\infty)\,|\beta\rangle\ , \tag{5.41}$$

where $\Omega(t)$ is related to the evolution operator by $U(t) \equiv \Omega^\dagger(t)\Omega(-\infty)$. Then the elements of the S-matrix are

$$\begin{aligned} \langle \Psi_{\rm out}(\beta) | \Psi_{\rm in}(\alpha) \rangle &= \lim_{t\to\infty} \left\langle \beta \left| \Omega^\dagger(t)\Omega(-\infty) \right| \alpha \right\rangle \\ &= \lim_{t\to\infty} \langle \beta \,|U(t)|\, \alpha \rangle \ , \end{aligned} \tag{5.42}$$

which are identical to the matrix elements of the S-operator defined above. There are certain assumptions involved in taking these limits, but for all purposes we can ignore such subtleties and work with the S-operator.

5.4 Wick's theorem

Each term in the perturbative expansion of the S-matrix contains a time-ordered product of a number of factors of $\mathscr{H}_I$, each of which is normal ordered. The normal ordering procedure involved putting all the annihilation operators to the right of all the creation operators so that it annihilates the vacuum. But the time ordering raises complications because in it all operators at earlier times must be further to the right. So creation operators at earlier times would be to the right of annihilation operators at later times, contrary to what we need for normal ordering. The advantage of normal ordered products is that their expectation values vanish in the vacuum. So we would

like to get back from time ordered products to normal ordered products. For this, we need a relation between the two. This relation is provided by *Wick's theorem.*

Consider first the simple example of the time-ordered product of two factors of a scalar field. Remember the definition given in Eq. (3.83), which can be written as

$$\mathscr{T}\left[\phi(x)\phi(x')\right] \equiv \Theta(t-t')\phi(x)\phi(x') + \Theta(t'-t)\phi(x')\phi(x)\,. \quad (5.43)$$

Our task reduces to writing the simple products on the right hand side in terms of normal ordered products. To this end, let us write

$$\phi(x) = \phi_+(x) + \phi_-(x)\,, \quad (5.44)$$

where the ϕ_+ contains the annihilation operator and the ϕ_- contains the creation operator. They are thus shorthands for the two terms appearing in Eq. (3.18). Because of the definition of the vacuum state in Eq. (3.32), we can write

$$\phi_+(x)\,|0\rangle = 0\,, \qquad \langle 0|\,\phi_-(x) = 0\,. \quad (5.45)$$

Now,

$$\begin{aligned}\phi(x)\phi(x') = {} & \phi_+(x)\phi_+(x') + \phi_+(x)\phi_-(x') \\ & +\phi_-(x)\phi_+(x') + \phi_-(x)\phi_-(x')\,. \end{aligned} \quad (5.46)$$

On the other hand, if we write the normal ordered operator $:\phi(x)\phi(x'):$, the only difference in the right side of Eq. (5.46) will be in the second term, where the order of the two operators must be reversed. In other words,

$$\begin{aligned} :\phi(x)\phi(x'): &= \phi(x)\phi(x') - \phi_+(x)\phi_-(x') + \phi_-(x')\phi_+(x) \\ &= \phi(x)\phi(x') - \left[\phi_+(x), \phi_-(x')\right]_-\,. \end{aligned} \quad (5.47)$$

The commutator on the right side can be put into a more convenient form. Because of Eq. (5.45), we have

$$\langle 0\,|\phi(x)\phi(x')|\,0\rangle = \langle 0\,|\phi_+(x)\phi_-(x')|\,0\rangle\,, \quad (5.48)$$

since the other terms vanish. We can also write it as

$$\langle 0\,|\phi(x)\phi(x')|\,0\rangle = \left\langle 0\right|\left[\phi_+(x), \phi_-(x')\right]_-\left|0\right\rangle\,, \quad (5.49)$$

since the extra term introduced in the process vanishes anyway owing to Eq. (5.45). However, since the commutator appearing on the right side is a number, its vacuum expectation value is the number itself. Thus we can write

$$\langle 0 \left| \phi(x)\phi(x') \right| 0 \rangle = [\phi_+(x), \phi_-(x')]_- \ . \tag{5.50}$$

Putting all this together, we have

$$\phi(x)\phi(x') = :\phi(x)\phi(x'): + \langle 0 \left| \phi(x)\phi(x') \right| 0 \rangle \ . \tag{5.51}$$

Putting this in the definition of the time-ordered product, Eq. (5.43), we obtain

$$\mathscr{T}\left[\phi(x)\phi(x')\right] = :\phi(x)\phi(x'): + \langle 0 \left| \mathscr{T}\left[\phi(x)\phi(x')\right] \right| 0 \rangle \ , \tag{5.52}$$

since $\Theta(t-t') + \Theta(t'-t) = 1$, and for scalar fields $:\phi(x)\phi(x'): = :\phi(x')\phi(x):$.

On the right side here, we have the vacuum expectation value of the time-ordered product of two field operators. Since such quantities will appear often in the ensuing discussion, it is useful to develop a compact notation for them. Such quantities are usually called *Wick contractions* and are denoted by the following symbol:

$$\langle 0 \left| \mathscr{T}\left[\phi(x)\phi(x')\right] \right| 0 \rangle \equiv \underbrace{\phi(x)\phi(x')} \ . \tag{5.53}$$

Thus, Eq. (5.52) can be rewritten with the help of this notation in the form

$$\mathscr{T}\left[\phi(x)\phi(x')\right] = :\phi(x)\phi(x'): + \underbrace{\phi(x)\phi(x')} \ . \tag{5.54}$$

More generally when the two fields appearing in the product are not necessarily the same, we will write

$$\mathscr{T}\left[\Phi(x)\Phi'(x')\right] = :\Phi(x)\Phi'(x'): + \underbrace{\Phi(x)\Phi'(x')} \ . \tag{5.55}$$

□ **Exercise 5.3** Show that for two fermionic operators $\psi(x)$ and $\psi'(x')$, although the definition of the time-ordered product has a negative sign for one of the terms, one still obtains Eq. (5.55) since the commutators also get replaced by anticommutators.

Since the contraction is a vacuum expectation value, it will vanish unless one operator (to the right) creates a particle which the other

operator (to the left) annihilates. So if they correspond to different fields, this expectation value vanishes. If they correspond to the same field (or a field and its adjoint) we get i times the respective Feynman propagators,

$$\begin{aligned}
\underbrace{\phi(x_1)\phi}(x_2) &= i\Delta_F(x_1 - x_2)\,, \\
\underbrace{\phi(x_1)\phi^\dagger}(x_2) &= \underbrace{\phi^\dagger(x_2)\phi}(x_1) = i\Delta_F(x_1 - x_2)\,, \\
\underbrace{\psi_\alpha(x_1)\overline{\psi}}_\beta(x_2) &= -\underbrace{\overline{\psi}_\beta(x_2)\psi}_\alpha(x_1) = iS_{F\alpha\beta}(x_1 - x_2)\,. \qquad (5.56)
\end{aligned}$$

Wick's theorem can be generalized to any number of field operators by multiplying on the right by a field and considering the interchanges required to change from a time ordered product to a normal ordered product. The theorem is then proven by induction. The result is

$$\begin{aligned}
&\mathscr{T}\,[\Phi_1(x_1)\cdots\Phi_n(x_n)] \\
&\quad = :\Phi_1\cdots\Phi_n: \\
&\quad + \left[:\underbrace{\Phi_1\Phi_2}\ \Phi_3\cdots\Phi_n: + \text{perm}\right] \\
&\quad + \left[:\underbrace{\Phi_1\Phi_2}\ \underbrace{\Phi_3\Phi_4}\ \Phi_5\cdots\Phi_n: + \text{perm}\right] \\
&\quad + \cdots\cdots \\
&\quad + \begin{cases} \left[\underbrace{\Phi_1\Phi_2}\ \cdots\underbrace{\Phi_{n-1}\Phi_n}\ + \text{perm}\right] & \text{if } n \text{ is even}, \\ \left[\underbrace{\Phi_1\Phi_2}\ \cdots\underbrace{\Phi_{n-2}\Phi_{n-1}}\ \Phi_n + \text{perm}\right] & \text{if } n \text{ is odd}. \end{cases} \qquad (5.57)
\end{aligned}$$

A few words about the notation used in writing up this result. All possible pairs are contracted, first one pair, then two pairs, and so on until we run out of pairs to contract. The word 'perm' occurring here means permutations on all fields present. The formula is long. So, in order to save space, we have omitted the space-time points corresponding to the field operators. It is to be understood that the field Φ_1 is a function of the space-time point x_1, and so on. The contractions are shown within the normal-ordered products. However, since the contractions are not field operators but rather the vacuum expectation values of some combinations of them, they should be treated as numbers and can therefore be written outside the normal-ordering symbol as well.

Finally, in the S-matrix expansion, we had time-ordered products of normal-ordered objects:

$$\mathscr{T}\,[\mathscr{H}_I(x_1)\cdots\mathscr{H}_I(x_n)] = \mathscr{T}\,[:AB\cdots(x_1):\cdots:AB\cdots(x_n):]. \quad (5.58)$$

We need to avoid commutators or anticommutators at the same point in space-time because they are infinite. For this, we change the times of creation and annihilation operators in the r-th factor by some small amount $\epsilon_r > 0$ so that creation operators act after annihilation operators,

$$\begin{aligned}\Phi_+(x_r) &\to \Phi_+(x_r^0 - \epsilon_r, \boldsymbol{x}_r)\,,\\ \Phi_-(x_r) &\to \Phi_-(x_r^0 + \epsilon_r, \boldsymbol{x}_r)\,.\end{aligned} \quad (5.59)$$

Then within each normal ordered group we have automatic time ordering so there is no contraction within a group. Then we take $\epsilon_r \to 0$ after we have done all other contractions, and we have the result

$$\begin{aligned}&\mathscr{T}\,[:AB\cdots(x_1):\cdots:AB\cdots(x_n):]\\ &\qquad = \mathscr{T}\,[(AB\cdots(x_1))\cdots(AB\cdots(x_n))]_{\text{no e.t.c.}}\,,\end{aligned} \quad (5.60)$$

where the right hand side is expanded according to Wick's theorem, but avoiding all equal-time contractions.

Wick's theorem looks formidable, but actually its implementation is not very difficult. In calculating S-matrix elements for specific initial and final states, we need to take the matrix elements of the time-ordered product given above between those states. Thus in the Wick expansion, we need only those terms where, apart from the contractions, we have just the right number of field operators to annihilate the initial state and create the final one. These comments will probably be clearer as we look at specific examples.

□ **Exercise 5.4** ⋆ Assuming that Wick's theorem holds for the product of n fields as in Eq. (5.57), show that it holds for the product of $n+1$ fields. The theorem is then proved by induction, starting from Eq. (5.55).

Chapter 6

From Wick expansion to Feynman diagrams

In this chapter, we show how to use the Wick expansion to calculate S-matrix elements involving scalars and spinors. As a concrete example, we take a theory with a fermion field and a scalar field, which interact via the Yukawa interaction:

$$\mathscr{L}_{\text{int}} = -h\bar{\psi}\psi\phi\,. \tag{6.1}$$

We will consider specific initial and final states, and in each case see which terms in the S-matrix expansion can give non-zero contribution to the relevant S-matrix element. An alternative approach could involve considering a specific order in the S-matrix expansion and investigate which processes it can contribute to. This approach will be taken later in the book for Quantum Electrodynamics.

6.1 Yukawa interaction : decay of a scalar

Let the quantum of the field ϕ be denoted by B, since the particle is a boson. The quanta of the fermionic field ψ will be called electrons. Of course, the mathematical results that follow will not depend on whether the quanta of ψ are electrons or some other spin-$\frac{1}{2}$ particle. We give the names only to avoid cumbersome sentences when we explain what is going on.

Let us denote the mass of the particle B by M and the mass of the electron by m. Suppose $M > 2m$, so that kinematically it is possible to have the B particle decay into an electron-positron pair.

Let us denote this process by

$$B(k) \to e^-(p) + e^+(p')\,, \tag{6.2}$$

where k, p, p' are the 4-momenta of the particles. Our objective is to calculate the S-matrix elements for this process.

The Hamiltonian derived from the Lagrangian will also have the usual free terms, minus the interaction Lagrangian. This is obviously true for all theories without derivative interactions. It also happens to be true if derivative interactions are present. For the fermion-scalar interaction,

$$\mathscr{H}_I = h :\overline{\psi}\psi\phi: \,. \tag{6.3}$$

Let us now look at the term linear in the interaction Hamiltonian in the S-matrix. It is

$$\begin{aligned} S^{(1)} &= \mathscr{T}\left[-i\int d^4x\, \mathscr{H}_I(x)\right] \\ &= -ih\int d^4x\, :\overline{\psi}\psi\phi: \,. \end{aligned} \tag{6.4}$$

In the last step, we have omitted the time-ordering sign, since in this case, there is only one space-time point involved and so there is just one time. Consequently, Wick's theorem is not needed here — we already have the normal ordered product.

It will be convenient at this point if we use the notation of Eq. (5.44) and a similar one for the ψ-field, where the creation and the annihilation parts of the field operator are denoted separately. Thus for the ψ-field, the ψ_+ part will contain the annihilation operator for the electron and the ψ_- part will contain the creation operator for the positron. Similarly, in $\overline{\psi}$, the part containing the annihilation operator for the positron will be denoted by $\overline{\psi}_+$, and the part containing the creation operator for the electron is $\overline{\psi}_-$. The subscript is what we get for the sign of E when we act on the Fourier transform by the energy operator $i\partial/\partial t$. A good mnemonic for this notation is that the subscripted sign on the field operator is the negative of the sign in the exponential factor appearing in the Fourier decomposition of the field operator. Using these notations, we can write

$$S^{(1)} = -ih\int d^4x\, :(\overline{\psi}_+ + \overline{\psi}_-)(\psi_+ + \psi_-)(\phi_+ + \phi_-): \,. \tag{6.5}$$

Consider now the matrix element of this operator between the initial and the final states given in Eq. (6.2). If we use the terms in $S^{(1)}$ which involve the ϕ_- and consider $\phi_- |B\rangle$, it will create another B-particle in the initial state, and this can never match the final state. Thus such terms will vanish. On the other hand, if we consider $\phi_+ |B\rangle$, the particle B of the initial state will be annihilated by the operator, and we will get the vacuum state. From the vacuum state, the operator $\overline{\psi}_-\psi_-$ can create an electron and a positron. Since this matches the final state, we will get a non-trivial result. In other words, among the eight terms occurring in the notation of Eq. (6.5), only one term will contribute to the matrix element of the process in question, which is

$$-ih \int d^4x \, \overline{\psi}_-\psi_-\phi_+ \, . \tag{6.6}$$

In writing this form, we have put the operators in the normal ordered form anyway, so we have omitted the normal-ordering symbol. All the field operators pertain to the space-time point x. Thus the annihilation of B occurs at the same point in space-time where the electron-positron pair is created. Pictorially, this can be represented by the diagram in Fig. 6.1.

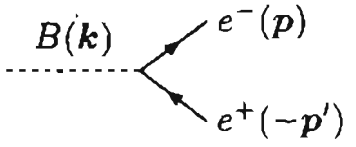


Figure 6.1: Feynman diagram for S-matrix element coming from Eq. (6.6).

This diagram is a trivial example of what are called *Feynman diagrams*. This one has three lines, representing the initial and final particles. They meet at a point, called a *vertex*, where the fields interact. In order to distinguish the scalar field from the fermions, we have used a dashed line for the scalar field and solid lines for the fermions.

Let us describe our convention of putting the arrows on the lines in these diagrams. Time will be taken to flow from left to right in all diagrams. Initial particles enter a diagram from the left, and final particles leave a diagram to the right.

The arrow on an electron line will always point to the future, i.e., to the right. The arrow on a positron line will always point to the past, i.e., to the left. Since momentum always points from the past to the future, it follows that for the electron, momentum flows along the arrow, whereas for the positron, momentum flows against the arrow. In addition, we have put a minus sign in the positron momentum. So, in the diagram, the positron appears to be going into the vertex with a momentum $-p'$, which means it is in fact leaving the vertex with a momentum p'.

The convention on time's direction varies from one book to another. In fact, if the lines are labeled as e^+ or e^-, the arrows are not needed to decide which one is electron and which one is positron. Then one can dispense with time's direction altogether and put the arrow in the direction of the momentum. However, an additional advantage of fixing a direction for the flow of time is that charge conservation is automatically ensured at each vertex if the arrow is made continuous. Therefore, and for ease of understanding, we shall allow a little redundancy and do both. We shall label the lines as e^+ or e^-, as well as take time to flow from left to right.

The value of the matrix element obtained from this term will be discussed in §6.3. For now, let us just try to see which other terms have non-zero contributions to the process of Eq. (6.2). Let us look at the second order term in the S-matrix expansion. Here we will have two factors of the interaction Hamiltonian. These factors will contain six field operators. Of them, we need three to annihilate the initial particle and create the final ones. That leaves us with three more. Since this is an odd number, all these remaining operators cannot be combined into contractions. In the Wick expansion, we will have terms which have one, or all three of these remaining operators sitting outside the contractions. When we take the matrix element, these terms will vanish. Extending this argument, one can show that the matrix elements for this process will vanish for all even powers of this interaction Hamiltonian.

The first non-trivial corrections to the amplitude obtained with $S^{(1)}$ appear in $S^{(3)}$, the term with three factors of the interaction

Hamiltonian. This term is

$$S^{(3)} = \frac{(-ih)^3}{3!} \int d^4x_1 \int d^4x_2 \int d^4x_3 \mathscr{T}\left[:(\overline{\psi}\psi\phi)_{x_1}: \ :(\overline{\psi}\psi\phi)_{x_2}: \ :(\overline{\psi}\psi\phi)_{x_3}: \right], \quad (6.7)$$

where we have indicated the space-time dependence of each normal ordered term by a subscript. This is done just to save some writing.

The Wick expansion of this term will now involve a lot of terms. Let us therefore start with some initial observations to help our way through the maze. First, we know from Eq. (5.57) that in the Wick expansion, we must take all permutations. However, all such terms are related by a renaming of the space-time points. Since all the interaction points are integrated over in Eq. (6.7), these terms are equal to one another. We can get rid of these repetitions by considering terms in the Wick expansion which are different irrespective of the labels assigned to the different space-time points. The permutation of the space-time points would then provide a factor of $n!$ for the n-th order term in the S-matrix expansion, which precisely cancels the $1/n!$ coming from the exponential of Eq. (5.37).

This is an observation which holds in general, for any interaction Hamiltonian and any process in question. Now let us turn to the process we have been discussing. Remember that we need one ϕ operator to annihilate the particle in the initial state, and a $\overline{\psi}$ and a ψ to create the particles in the final state. All the other fields must be contracted, otherwise the matrix element will vanish. Thus, in the Wick expansion of $S^{(3)}$, the only terms which will give a non-zero matrix element for the process of our interest are the ones which have three pairs of contracted fields. The contraction of fields of different types, e.g., ϕ with ψ or $\overline{\psi}$, or ψ with ψ, vanish. Let us denote the relevant terms with a superscript indicating the number of contractions and write them out:

$$\mathscr{T}\left[:(\overline{\psi}\psi\phi)_{x_1}: \ :(\overline{\psi}\psi\phi)_{x_2}: \ :(\overline{\psi}\psi\phi)_{x_3}: \right]^{(3)} = :\underbrace{(\overline{\psi}\psi\phi)_{x_1}}\ \underbrace{(\overline{\psi}\psi\phi)_{x_2}}\ \underbrace{(\overline{\psi}\psi\phi)_{x_3}}: + :\underbrace{(\overline{\psi}\psi\phi)_{x_1}}\ \underbrace{(\overline{\psi}\psi\phi)_{x_2}}\ \underbrace{(\overline{\psi}\psi\phi)_{x_3}}:$$

$$
\begin{aligned}
&+ :(\overline{\psi}\psi\phi)_{x_1}\ (\overline{\psi}\psi\phi)_{x_2}\ (\overline{\psi}\psi\phi)_{x_3}: \\
&+ :(\overline{\psi}\psi\phi)_{x_1}\ (\overline{\psi}\psi\phi)_{x_2}\ (\overline{\psi}\psi\phi)_{x_3}: \\
&+ :(\overline{\psi}\psi\phi)_{x_1}\ (\overline{\psi}\psi\phi)_{x_2}\ (\overline{\psi}\psi\phi)_{x_3}: \qquad (6.8)
\end{aligned}
$$

We can represent all such terms pictorially by Feynman diagrams. Consider the first term on the right hand side of Eq. (6.8). Let us rewrite it by separating out the contractions. This time, we explicitly write the space-time point corresponding to each operator, and also the spinor indices:

$$
-\phi(x_2)\phi(x_3)\ \overline{\psi}_\alpha(x_1)\psi_\beta(x_2)\ \psi_\alpha(x_1)\overline{\psi}_\gamma(x_3)\ :\overline{\psi}_\beta(x_2)\psi_\gamma(x_3)\phi(x_1): . \qquad (6.9)
$$

The minus sign appears because of the rearrangement of the fermion field operators. It is an odd permutation of the original order, hence the sign. We can interchange the order of another pair of fermion field operators to rewrite it as

$$
+\phi(x_2)\phi(x_3)\ \psi_\beta(x_2)\overline{\psi}_\alpha(x_1)\ \psi_\alpha(x_1)\overline{\psi}_\gamma(x_3)\ :\overline{\psi}_\beta(x_2)\psi_\gamma(x_3)\phi(x_1): . \qquad (6.10)
$$

In the normal-ordered factor, the only term which gives a non-zero matrix element for the process in question is given by

$$
\overline{\psi}^{\beta}_{-}(x_2)\psi^{\gamma}_{-}(x_3)\phi_{+}(x_1)\,, \qquad (6.11)
$$

as we argued just before Eq. (6.6). The spinor indices have been raised only for convenience, and are equivalent to lower indices. Thus the boson B is annihilated at the point x_1, the electron is created at x_2. and the positron is created at x_3. Also, from Eqs. (6.10) and (5.56), we find that there is a scalar propagator between x_2 and x_3, and two fermion propagators, one from x_1 to x_2, and another from x_3 to x_1. All these statements can be pictorially represented by the Feynman diagram of Fig. 6.2a.

Notice that the contractions now appear as *internal lines* of this diagram. By this, we mean lines which do not represent the particles in the initial state or the final state. For example, the contraction of $\phi(x_2)\phi(x_3)$ is now represented by a dashed line which runs between x_2

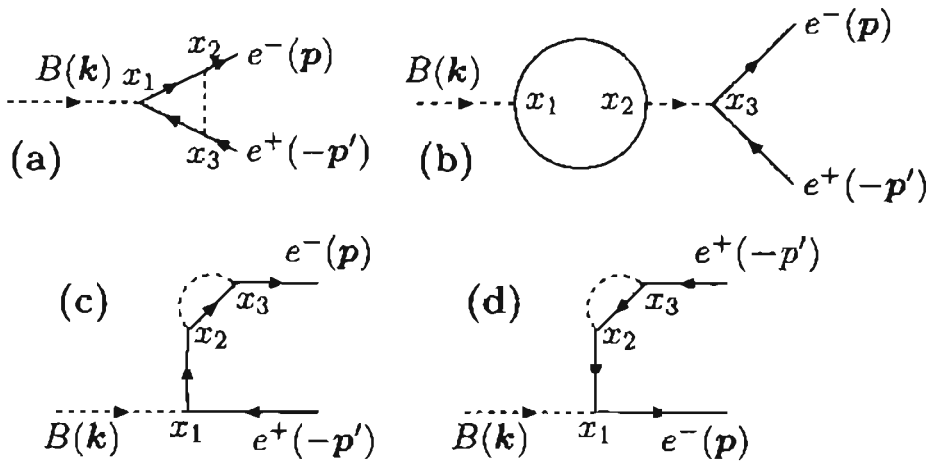


Figure 6.2: Feynman diagrams representing the first four terms of Eq. (6.8).

and x_3. Similarly, there are two internal fermion lines, one between x_1 and x_2, and another between x_3 and x_1. These internal lines appear as propagators in the evaluation of the S-matrix element.

An internal scalar line represents the contraction between two ϕ-operators at two space-time points. For an internal fermion line, however, the situation is slightly different, because here the contraction is between the ψ-operator at one point and the $\overline{\psi}$-operator at another. As we know, the propagator represents a particle being created at one space-time point and annihilated at another. We can indicate the points of creation and annihilation in the diagram itself, by putting an arrow on the internal line which points towards the end where the ψ-operator occurs. For example, from Eq. (6.10), we see that for the fermion line between x_1 and x_2, we have the ψ-operator at x_2 and the $\overline{\psi}$-operator at x_1. Consequently, we have put an arrow pointing from x_1 to x_2 on the fermion line in Fig. 6.2a. Similarly, for the other internal fermion line in this diagram, the direction of the arrow is from x_3 to x_1. This rule automatically maintains the continuity of arrows on the fermion lines.

Let us now move on to the next term in Eq. (6.8). Here we have an internal fermion line going from x_1 to x_2, but there is also one coming back from x_2 to x_1. Therefore, the Feynman diagram has a *fermion loop* as shown in Fig. 6.2b. We also find a B propagator between x_2 and x_3. Note that the arrow convention is valid only for the initial and final particles — internal particles may be neither in the past nor in the future of one another.

In the third term of Eq. (6.8), the initial B is annihilated at x_1, which is also the point where the final e^+ has been created. The Feynman diagram for this case has been given in Fig. 6.2c. The fourth term of Eq. (6.8) is equivalent to the diagram in Fig. 6.2d, since here the final e^- has been created at the point where the initial B has been annihilated.

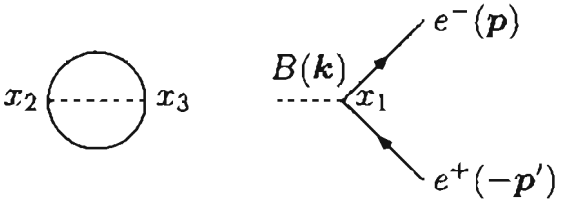


Figure 6.3: Feynman diagram representing the last term of Eq. (6.8).

The last term remains to be discussed. It is pictorially represented in Fig. 6.3. As is clear from the diagram, this term represents the amplitude of Fig. 6.1 times an amplitude for a vacuum to vacuum transition. This latter part comes from the loop shown, since it has no external legs. If the vacuum is stable, as it is assumed to be for all physical systems, all such vacuum to vacuum amplitudes must add up to an overall phase factor. Since such diagrams can be added to any non-trivial diagram, they provide the same overall phase factor to all diagrams. Therefore, we will happily ignore all vacuum to vacuum processes.

□ **Exercise 6.1** Draw the Feynman diagram(s) at lowest order at which the scalar-scalar scattering $BB \to BB$ takes place.

□ **Exercise 6.2** Write down the hermitian conjugate term in Eq. (5.8), and show that it can mediate processes like $p + e \to n + \nu_e$.

6.2 Normalized states

So far, we have shown how only a few terms in the Wick expansion contribute to the S-matrix elements for specified initial and final states. We have not calculated any matrix element explicitly.

To perform such a calculation, we face a small problem. We have defined the states in the Fock space by acting on the vacuum by the creation operator, as for example in Eq. (3.39). However, this

definition is not the best for calculations, for two reasons. First, the creation operators have a mass dimension of $-3/2$, as can be checked from the Fourier decomposition of Eq. (3.18) for the scalar fields, and of Eq. (4.96) for the spinor fields. Thus with the definition of Eq. (3.39), the one-particle states have different dimensions from the vacuum state. Continuing on this argument, the dimension of the two-particle states will be more different, and so on for other many particle states. The other problem is that the norm of any one-particle state diverges, as seen from Eq. (3.40) if we put $p = p'$.

To avoid these problems, we can define the states within a finite region of volume V. Physically, this does not create any problem provided the volume V is large enough so that the boundaries of this region are far removed from the region of interaction. We can calculate quantities of physical interest using such states, and then take the limit $V \to \infty$ at the end of the calculations. With this in mind, we continue to use the Fourier expansions of fields in the infinite volume limit.

The vacuum state is assumed to be dimensionless by virtue of the normalization condition in Eq. (3.33). We want to define the other states to be dimensionless as well. Let us now define the one-particle states as

$$|B(\boldsymbol{p})\rangle \equiv \sqrt{\frac{(2\pi)^3}{V}}\, a^\dagger(\boldsymbol{p})\,|0\rangle \tag{6.12}$$

for the scalar field whose quanta are denoted by B. Note that the ket is now dimensionless. Similarly, for the electrons and the positrons, we define the one-particle states with momentum $\boldsymbol{p}$ and spin s as

$$\begin{aligned} |e^-(\boldsymbol{p},s)\rangle &\equiv \sqrt{\frac{(2\pi)^3}{V}}\, f_s^\dagger(\boldsymbol{p})\,|0\rangle\ , \\ |e^+(\boldsymbol{p},s)\rangle &\equiv \sqrt{\frac{(2\pi)^3}{V}}\, \widehat{f}_s^\dagger(\boldsymbol{p})\,|0\rangle\ . \end{aligned} \tag{6.13}$$

Using the commutation relations, our states are then normalized as

$$\begin{aligned} \langle B(\boldsymbol{p})\,|B(\boldsymbol{p}')\rangle &= \frac{(2\pi)^3}{V}\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \\ \langle e^-(\boldsymbol{p},s)\,|e^-(\boldsymbol{p}',s')\rangle &= \frac{(2\pi)^3}{V}\delta_{ss'}\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,, \\ \langle e^+(\boldsymbol{p},s)\,|e^+(\boldsymbol{p}',s')\rangle &= \frac{(2\pi)^3}{V}\delta_{ss'}\delta^3(\boldsymbol{p}-\boldsymbol{p}')\,. \end{aligned} \tag{6.14}$$

In all these relations, we have only used the 3-momentum of the particles concerned, without making any reference to the energy. The reason is that these are physical states which must obey the relation $p^2 = m^2$, so the energy is determined by the 3-momentum.

We have solved the problem of the dimension of the states, but the other problem still remains. For example,

$$\langle B(\boldsymbol{p}) | B(\boldsymbol{p}) \rangle = \frac{(2\pi)^3}{V} \delta^3(\boldsymbol{0})_p \,, \tag{6.15}$$

which is meaningless. However, this is not unexpected. This $\delta^3(\boldsymbol{0})_p$ appears in all problems involving normalization of plane waves. The reason is that plane waves are spread over infinite volumes and time intervals. This is not a problem specifically with quantum field theory, but with plane waves in any theory. We need to carefully eliminate such infinities from all physical quantities.

The way to solve this problem is to restrict the plane waves to a box and then to let the box become infinite in size. Then,

$$\delta^3(\boldsymbol{p}) = \lim_{V \to \infty} \left(\frac{1}{(2\pi)^3} \int_V d^3x \, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \,, \tag{6.16}$$

where the integration is done over the volume V, as indicated by the subscript after the integration sign. Before taking the limit of V going to infinity, we obtain

$$\delta^3(\boldsymbol{0})_p = \frac{V}{(2\pi)^3} \,. \tag{6.17}$$

With this definition, we see that the one-particle states of Eqs. (6.12) and (6.13) are normalized to unity, provided we take the limit of $V \to \infty$ at the end of the calculation of any physical quantity.

Now we can write down the action of various field operators on different one-particle states:

$$\begin{aligned}
\phi_+(x) | B(\boldsymbol{k}) \rangle &= \frac{1}{\sqrt{2\omega_k V}} e^{-ik\cdot x} |0\rangle \,, \\
\psi_+(x) | e^-(\boldsymbol{p}, s) \rangle &= \frac{1}{\sqrt{2E_p V}} u_s(\boldsymbol{p}) e^{-ip\cdot x} |0\rangle \,, \\
\overline{\psi}_+(x) | e^+(\boldsymbol{p}', s') \rangle &= \frac{1}{\sqrt{2E_{p'} V}} \overline{v}_{s'}(\boldsymbol{p}') e^{-ip'\cdot x} |0\rangle \,,
\end{aligned} \tag{6.18}$$

where ω_k and E_p represent the energies of the scalar and the electron for the 3-momenta in the subscripts.

Similarly, for the adjoint operators

$$\begin{aligned}
\langle B(\boldsymbol{k})|\,\phi_-(x) &= \frac{1}{\sqrt{2\omega_k V}} e^{ik\cdot x}\,\langle 0|\ , \\
\langle e^-(\boldsymbol{p},s)|\,\overline{\psi}_-(x) &= \frac{1}{\sqrt{2E_p V}}\,\overline{u}_s(\boldsymbol{p}) e^{ip\cdot x}\,\langle 0|\ , \\
\langle e^+(\boldsymbol{p}',s')|\,\psi_-(x) &= \frac{1}{\sqrt{2E_{p'} V}}\,v_{s'}(\boldsymbol{p}') e^{ip'\cdot x}\,\langle 0|\ ,
\end{aligned} \tag{6.19}$$

These results have to be used quite often when we try to evaluate the S-matrix elements, as we now show.

□ **Exercise 6.3** Prove the results of Eq. (6.18), using the Fourier decomposition of the fields and the commutation (or anticommutation) relations obeyed by the creation and annihilation operators.

6.3 Sample calculation of a matrix element

Consider first the process of Eq. (6.2), where a B-particle decays to an electron-positron pair. We have already argued that in the lowest order in the expansion of the S-matrix the only term which contributes to this matrix element is the term shown in Eq. (6.6). The matrix element of this term between the initial and the final state is then

$$S_{fi}^{(1)} = -ih \int d^4x\ \left\langle e^-(p) e^+(p') \left| \overline{\psi}_- \psi_- \phi_+ \right| B(k) \right\rangle\ . \tag{6.20}$$

Using the relations in Eqs. (6.18) and (6.19), we obtain

$$\begin{aligned}
S_{fi}^{(1)} &= (-ih)\overline{u}_s(\boldsymbol{p}) v_{s'}(\boldsymbol{p}') \\
&\quad \times \int d^4x\, e^{i(p+p'-k)\cdot x} \frac{1}{\sqrt{2\omega_k V}} \frac{1}{\sqrt{2E_p V}} \frac{1}{\sqrt{2E_{p'} V}}\ ,
\end{aligned} \tag{6.21}$$

assuming that the spin of the final electron is s and that of the positron is s', and also using the fact that the vacuum state is normalized to unity.

Consider now the integration over x. In the infinite volume limit, we can write

$$\int d^4x\, e^{i(p+p'-k)\cdot x} = (2\pi)^4 \delta^4(k - p - p')\,. \tag{6.22}$$

Inserting this in the expression obtained before, we obtain

$$S_{fi}^{(1)} = (-ih)\left[\overline{u}_s(\boldsymbol{p})v_{s'}(\boldsymbol{p}')\right]\left[(2\pi)^4\delta^4(k-p-p')\right] \times \left[\frac{1}{\sqrt{2\omega_k V}}\frac{1}{\sqrt{2E_p V}}\frac{1}{\sqrt{2E_{p'}V}}\right]. \qquad (6.23)$$

The δ-function thus obtained merely gives us the constraint from 4-momentum conservation. In other words, it tells us that the initial 4-momentum must be equal to the final 4-momentum.

To get a feeling for a more non-trivial case, let us now move on to the diagram given in Fig. 6.2a. The term in the Wick expansion that corresponds to this diagram was written down in Eq. (6.10). The contractions given in that formula are nothing but the vacuum expectation values of the time-ordered products of the operators involved, and therefore they should be replaced by i times the propagator of the particle involved. In other words, we should make the replacements

$$\begin{aligned} \phi(x_2)\phi(x_3) &= i\Delta_F(x_2-x_3) \\ \psi(x_1)\overline{\psi}(x_2) &= iS_F(x_1-x_2) \end{aligned} \qquad (6.24)$$

etc, where the propagators have been defined for the scalar fields in §3.7 and for spinor fields in §4.7. The contribution of the S-matrix element coming from this diagram can be written as

$$\begin{aligned} S_{fi}^{(3a)} = (-ih)^3 \int d^4x_1 \int d^4x_2 \int d^4x_3 \\ \times i\Delta_F(x_2-x_3)\, iS_{F\beta\alpha}(x_2-x_1) iS_{F\alpha\gamma}(x_1-x_3) \\ \times \left\langle e^-(p)e^+(p')\left|\overline{\psi}_-^{\beta}(x_2)\psi_-^{\gamma}(x_3)\phi_+(x_1)\right|B(k)\right\rangle . \end{aligned} \qquad (6.25)$$

On the left, we have put the superscript '3' to indicate that it is a term from the 3rd order expansion of the S-matrix, and the superscript 'a' to denote the particular diagram of Fig. 6.2.

We can use the Fourier transform of the propagators, and also

use the matrix elements derived in Eqs. (6.19) and (6.18) to write

$$\begin{aligned} S_{fi}^{(3a)} = (-ih)^3 \int d^4x_1 \int d^4x_2 \int d^4x_3 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4q_3}{(2\pi)^4} \\ \times i\Delta_F(q_1)\, e^{-iq_1\cdot(x_2-x_3)} \\ \times iS_{F\beta\alpha}(q_2)\, e^{-iq_2\cdot(x_2-x_1)}\, iS_{F\alpha\gamma}(q_3)\, e^{-iq_3\cdot(x_1-x_3)} \\ \times \left[\frac{e^{-ik\cdot x_1}}{\sqrt{2\omega_k V}} \frac{\overline{u}_s^{\beta}(\boldsymbol{p})\, e^{ip\cdot x_2}}{\sqrt{2E_p V}} \frac{v_{s'}^{\gamma}(\boldsymbol{p}')\, e^{ip'\cdot x_3}}{\sqrt{2E_{p'} V}}\right] . \end{aligned} \tag{6.26}$$

As in the case with $S^{(1)}$, we can perform the integrations on the space-time points. For example, the integral over all the factors which depend on x_1 gives

$$\int d^4x_1\, e^{iq_2\cdot x_1} e^{-iq_3\cdot x_1}\, e^{-ik\cdot x_1} = (2\pi)^4\delta^4(k-q_2+q_3)\,. \tag{6.27}$$

Similarly, integration over x_2 gives $(2\pi)^4\delta^4(q_1+q_2-p)$, and integration over x_3 gives $(2\pi)^4\delta^4(q_1+q_3+p')$.

If the arguments of all these δ-functions vanish, it automatically implies $k = p + p'$, which is the condition for overall momentum conservation. Thus we can rewrite the product of the δ-functions as

$$\begin{aligned} &\delta^4(k-q_2+q_3)\delta^4(q_1+q_2-p)\delta^4(q_1+q_3+p') \\ &\qquad = \delta^4(k-q_2+q_3)\delta^4(q_1+q_2-p)\delta^4(k-p-p')\,, \end{aligned} \tag{6.28}$$

since both expressions imply the vanishing of the same quantities. Inserting the expression in Eq. (6.26) and arranging the terms in a manner that the spinor indices need not be written down, we obtain

$$\begin{aligned} S_{fi}^{(3a)} = (-ih)^3 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4q_3}{(2\pi)^4} \\ \times (2\pi)^{12}\delta^4(k-q_2+q_3)\delta^4(q_1+q_2-p)\delta^4(k-p-p') \\ \times i\Delta_F(q_1)\,[\overline{u}_s(\boldsymbol{p})iS_F(q_2)iS_F(q_3)v_{s'}(\boldsymbol{p}')] \\ \times \left[\frac{1}{\sqrt{2\omega_k V}} \frac{1}{\sqrt{2E_p V}} \frac{1}{\sqrt{2E_{p'} V}}\right] . \end{aligned} \tag{6.29}$$

At this point, we can perform the integrations over q_2 and q_3 easily, and write q instead of the remaining unknown momentum q_1 for the

sake of brevity. This gives

$$S_{fi}^{(3a)} = (-ih)^3\,(2\pi)^4\delta^4(k-p-p')\int\frac{d^4q}{(2\pi)^4} \times i\Delta_F(q)\left[\overline{u}_s(\boldsymbol{p})iS_F(p-q)\,iS_F(p-q-k)v_{s'}(\boldsymbol{p}')\right] \times\left[\frac{1}{\sqrt{2\omega_k V}}\frac{1}{\sqrt{2E_pV}}\frac{1}{\sqrt{2E_{p'}V}}\right]. \tag{6.30}$$

This is the final expression. Notice that it involves the integration over an unknown momentum q. This is because the momenta of all the internal lines are not determined by the external momenta. In this diagram, there is one loop of internal lines, which admits one unknown momentum which must be integrated over. In general, if there are n loops, there should be n such unknown momenta which have to be integrated over. The diagrams without any loops are called *tree diagrams*, whose example is the first order contribution to the S-matrix discussed earlier. Such diagrams do not have any integration over unknown momenta, as seen in Eq. (6.23) above.

To appreciate another subtlety, let us try to find out the contribution to diagram Fig. 6.2b to the S-matrix element. The contractions were shown in Eq. (6.8). Written out with explicit indices for space-time points and spinor components, this term reads

$$\underbrace{\phi(x_2)}\phi(x_3)\;\underbrace{\overline{\psi}_\alpha(x_1)}\psi_\beta(x_2)\;\underbrace{\psi_\alpha(x_1)}\overline{\psi}_\beta(x_2)\;\;:\overline{\psi}_\gamma(x_3)\psi_\gamma(x_3)\phi(x_1):\,. \tag{6.31}$$

Let us now concentrate on the fermion contractions. We can exchange the order of the fermion fields in one of the contractions to get

$$\begin{aligned}-\underbrace{\psi_\beta(x_2)}\overline{\psi}_\alpha(x_1)\;\underbrace{\psi_\alpha(x_1)}\overline{\psi}_\beta(x_2) &= -iS_{F\beta\alpha}(x_2-x_1)\,iS_{F\alpha\beta}(x_1-x_2)\\ &= -\mathrm{Tr}\left[iS_F(x_2-x_1)\,iS_F(x_1-x_2)\right].\end{aligned} \tag{6.32}$$

This shows that if the diagram has a fermion line which closes on itself, the corresponding momentum-space amplitude should have a minus sign for the loop *and* a trace over the Dirac indices running in the loop.

□ **Exercise 6.4** Find the contributions of diagrams Fig. 6.2c and d to the S-matrix element. Leave your expressions at the stage where they involve integration over an unknown momentum. (In other words, do not integrate, or write the explicit forms for the propagators.)

6.4 Another example: fermion scattering

We now consider some non-trivialities which might arise if the initial or the final state (or both) contain some identical particles. As a concrete example, let us think of the process

$$e^-(p_1) + e^-(p_2) \to e^-(p_1') + e^-(p_2') \tag{6.33}$$

governed by the Yukawa interaction that we have been discussing.

This process will have no contribution at the first order in the S-matrix expansion, but will have contributions at the second order. To find the amplitude of these second order contributions, let us start by writing the relevant term of the S-matrix, which is:

$$S^{(2)} = \frac{(-ih)^2}{2!} \int d^4x_1 \int d^4x_2 \, \mathscr{T}\left[:(\overline{\psi}\psi\phi)_{x_1}: \; :(\overline{\psi}\psi\phi)_{x_2}: \right]. \tag{6.34}$$

After applying Wick's theorem, we will obtain a number of terms. For the process at hand, we want terms where four fermionic operators are not contracted, corresponding to the particles in the initial and the final states. There is only one such term, which can be written as

$$\begin{aligned} &\mathscr{T}\left[:(\overline{\psi}\psi\phi)_{x_1}: \; :(\overline{\psi}\psi\phi)_{x_2}: \right]^{(1)} \\ &\qquad = \; :\underline{(\overline{\psi}\psi\phi)_{x_1}} \; \underline{(\overline{\psi}\psi\phi)_{x_2}}: \\ &\qquad = \; \underline{\phi(x_1)}\phi(x_2) \; :\overline{\psi}^\alpha(x_1)\psi^\alpha(x_1) \; \overline{\psi}^\beta(x_2)\psi^\beta(x_2): \, , \end{aligned} \tag{6.35}$$

where α, β are spinor indices as usual. We have only electrons in the initial and the final states. Therefore, if we decompose ψ into ψ_+ and ψ_-, as in Eq. (6.5), the part which will contribute to the present process is

$$-\underline{\phi(x_1)}\phi(x_2) \; \overline{\psi}_-^\alpha(x_1)\overline{\psi}_-^\beta(x_2)\psi_+^\alpha(x_1)\psi_+^\beta(x_2) \, , \tag{6.36}$$

where now we have put the field operators in the normal order and omitted the normal ordering symbol, picking up a minus sign in the process. Putting this back in Eq. (6.34), we see that the S-matrix element between the initial and the final state is given by

$$S_{fi}^{(2)} = -\frac{(-ih)^2}{2!}\int d^4x_1 \int d^4x_2\, i\Delta_F(x_1 - x_2) \times \left\langle e^-(\boldsymbol{p}_1')e^-(\boldsymbol{p}_2') \left| \overline{\psi}_-^\alpha(x_1)\overline{\psi}_-^\beta(x_2)\psi_+^\alpha(x_1)\psi_+^\beta(x_2) \right| e^-(\boldsymbol{p}_1)e^-(\boldsymbol{p}_2) \right\rangle . \tag{6.37}$$

□ **Exercise 6.5** Argue that odd order terms in the S-matrix expansion cannot contribute to the process of Eq. (6.33).

Consider now the initial state. Using Eq. (6.13), we can write it as

$$|e^-(\boldsymbol{p}_1)e^-(\boldsymbol{p}_2)\rangle = \frac{(2\pi)^3}{V} f^\dagger(\boldsymbol{p}_2)f^\dagger(\boldsymbol{p}_1)\,|0\rangle\ . \tag{6.38}$$

Therefore,

$$\psi_+^\alpha(x_1)\psi_+^\beta(x_2)\,|e^-(\boldsymbol{p}_1)e^-(\boldsymbol{p}_2)\rangle = \int \frac{d^3k}{\sqrt{2E_k V}} \int \frac{d^3k'}{\sqrt{2E_{k'}V}}\, u^\alpha(k)u^\beta(k')e^{-ik\cdot x_1}e^{-ik'\cdot x_2} \times f(\boldsymbol{k})f(\boldsymbol{k}')f^\dagger(\boldsymbol{p}_2)f^\dagger(\boldsymbol{p}_1)\,|0\rangle\ . \tag{6.39}$$

We now use the identity

$$[AB, CD]_- = A[B,C]_+D - [A,C]_+BD + CA[B,D]_+ - C[A,D]_+B\,, \tag{6.40}$$

which holds for any four operators A, B, C and D. In the present case, using the anticommutation relations of Eq. (4.103), we obtain

$$\left[f(\boldsymbol{k})f(\boldsymbol{k}'), f^\dagger(\boldsymbol{p}_2)f^\dagger(\boldsymbol{p}_1)\right]_- = \delta^3(\boldsymbol{k}' - \boldsymbol{p}_2)f(\boldsymbol{k})f^\dagger(\boldsymbol{p}_1) - \delta^3(\boldsymbol{k} - \boldsymbol{p}_2)f(\boldsymbol{k}')f^\dagger(\boldsymbol{p}_1) + \delta^3(\boldsymbol{k}' - \boldsymbol{p}_1)f^\dagger(\boldsymbol{p}_2)f(\boldsymbol{k}) - \delta^3(\boldsymbol{k} - \boldsymbol{p}_1)f^\dagger(\boldsymbol{p}_2)f(\boldsymbol{k}')\,. \tag{6.41}$$

The important point is that the combination of creation and annihilation operators operating on the vacuum state in Eq. (6.39) can be replaced by the left hand side of Eq. (6.41), since the extra term

in the commutator annihilates the vacuum anyway. Now, looking at the right hand side of Eq. (6.41), we find that the last two terms annihilate the vacuum. As for the other two, we can use the anti-commutation relations once again and conclude that

$$f(\boldsymbol{k})f(\boldsymbol{k}')f^\dagger(\boldsymbol{p}_2)f^\dagger(\boldsymbol{p}_1)\,|0\rangle \\ = \Big(\delta^3(\boldsymbol{k}'-\boldsymbol{p}_2)\delta^3(\boldsymbol{k}-\boldsymbol{p}_1) - \delta^3(\boldsymbol{k}-\boldsymbol{p}_2)\delta^3(\boldsymbol{k}'-\boldsymbol{p}_1)\Big)\,|0\rangle\,. \quad (6.42)$$

Putting this back in Eq. (6.39) and performing the integrations over $\boldsymbol{k}$ and $\boldsymbol{k}'$, we obtain

$$\psi_+^\alpha(x_1)\psi_+^\beta(x_2)\,|e^-(\boldsymbol{p}_1)e^-(\boldsymbol{p}_2)\rangle \\ = \frac{1}{\sqrt{2E_1V}}\,\frac{1}{\sqrt{2E_2V}}\Big[u^\alpha(p_1)u^\beta(p_2)e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2} \\ - u^\alpha(p_2)u^\beta(p_1)e^{-ip_2\cdot x_1}e^{-ip_1\cdot x_2}\Big]\,|0\rangle\,, \quad (6.43)$$

where $E_1 = \sqrt{\boldsymbol{p}_1^2+m^2}$ etc. We now look at the final state, which is

$$\langle e^-(\boldsymbol{p}_1')e^-(\boldsymbol{p}_2')| = \frac{(2\pi)^3}{V}\,\langle 0|\,f(\boldsymbol{p}_2')f(\boldsymbol{p}_1')\,. \quad (6.44)$$

Following similar steps, we find

$$\langle e^-(\boldsymbol{p}_1')e^-(\boldsymbol{p}_2')|\,\overline{\psi}_-^\alpha(x_1)\overline{\psi}_-^\beta(x_2) \\ = \frac{1}{\sqrt{2E_1'V}}\,\frac{1}{\sqrt{2E_2'V}}\,\langle 0\,\Big[\overline{u}^\alpha(p_1')\overline{u}^\beta(p_2')e^{ip_1'\cdot x_1}e^{ip_2'\cdot x_2} \\ - \overline{u}^\alpha(p_2')\overline{u}^\beta(p_1')e^{ip_2'\cdot x_1}e^{ip_1'\cdot x_2}\Big]\,. \quad (6.45)$$

Going back to Eq. (6.37), we now rewrite the S-matrix element in the form

$$S_{fi}^{(2)} = -\frac{(-ih)^2}{2!}\int d^4x_1\int d^4x_2\int\frac{d^4q}{(2\pi)^4}\,i\Delta_F(q)e^{iq\cdot(x_1-x_2)} \\ \times\Big[\overline{u}^\alpha(p_1')\overline{u}^\beta(p_2')e^{ip_1'\cdot x_1}e^{ip_2'\cdot x_2} - \overline{u}^\alpha(p_2')\overline{u}^\beta(p_1')e^{ip_2'\cdot x_1}e^{ip_1'\cdot x_2}\Big] \\ \times\Big[u^\alpha(p_1)u^\beta(p_2)e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2} - u^\alpha(p_2)u^\beta(p_1)e^{-ip_2\cdot x_1}e^{-ip_1\cdot x_2}\Big] \\ \times\left[\frac{1}{\sqrt{2E_1V}}\,\frac{1}{\sqrt{2E_2V}}\,\frac{1}{\sqrt{2E_1'V}}\,\frac{1}{\sqrt{2E_2'V}}\right] \quad (6.46)$$

It seems that there are four different terms coming from multiplying the spinor factors. But since x_1 and x_2 are integrated over, we can interchange their labels and rewrite the expression as

$$\begin{aligned} S_{fi}^{(2)} &= (-ih)^2 \int d^4x_1 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4}\, i\Delta_F(q) e^{iq\cdot(x_1-x_2)} \\ &\quad \times \left[\overline{u}^\alpha(p_2')\overline{u}^\beta(p_1') e^{ip_2'\cdot x_1} e^{ip_1'\cdot x_2} - \overline{u}^\alpha(p_1')\overline{u}^\beta(p_2') e^{ip_1'\cdot x_1} e^{ip_2'\cdot x_2}\right] \\ &\quad \times u^\alpha(p_1) u^\beta(p_2) e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_2} \\ &\quad \times \left[\frac{1}{\sqrt{2E_1V}} \frac{1}{\sqrt{2E_2V}} \frac{1}{\sqrt{2E_1'V}} \frac{1}{\sqrt{2E_2'V}}\right]. \end{aligned} \tag{6.47}$$

Here we have used $\Delta_F(q) = \Delta_F(-q)$.

We can now perform the integrations over x_1 and x_2, which will give two momentum conserving δ-functions in each term. We can rearrange the arguments of these δ-functions as we did in Eq. (6.28) such that one of the arguments becomes independent of the internal momentum q. With the other one, we can perform the q-integration trivially. After these steps, we obtain

$$\begin{aligned} S_{fi}^{(2)} &= (-ih)^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \\ &\quad \times \left[\overline{u}^\alpha(p_2')\overline{u}^\beta(p_1') i\Delta_F(p_1 - p_2') - \overline{u}^\alpha(p_1')\overline{u}^\beta(p_2') i\Delta_F(p_1 - p_1')\right] \\ &\quad \times u^\alpha(p_1) u^\beta(p_2) \\ &\quad \times \left[\frac{1}{\sqrt{2E_1V}} \frac{1}{\sqrt{2E_2V}} \frac{1}{\sqrt{2E_1'V}} \frac{1}{\sqrt{2E_2'V}}\right]. \end{aligned} \tag{6.48}$$

We can make two observations. First, although we started from only one term in the Wick expansion, we ended up with two terms in the

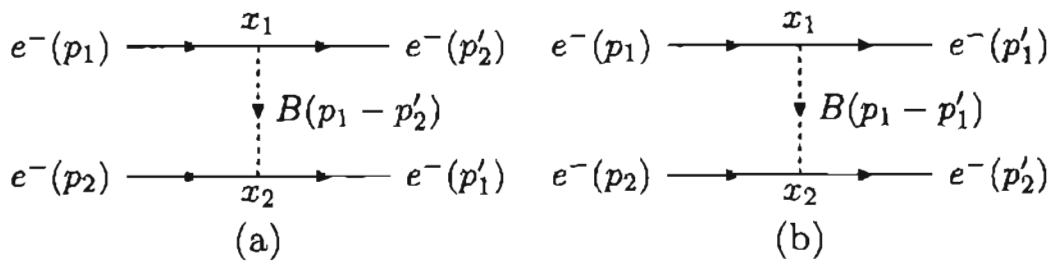


Figure 6.4: Lowest order diagrams for electron-electron scattering in the Yukawa theory.

S-matrix element, because the initial and the final states now have two identical particles. These two terms correspond to the Feynman diagrams of Fig. 6.4.

The second point is that the second term in Eq. (6.48) can be obtained by interchanging p_1' and p_2' in the first term. But this term comes with the opposite sign. This is expected since if two identical fermions are interchanged, the quantum mechanical amplitude changes sign.

6.5 Feynman amplitude

In the expression for various S-matrix elements calculated in §6.3 and §6.4, we can identify three types of factors. One of these is a δ-function for the conservation of 4-momentum in the overall process, which is multiplied by $(2\pi)^4$. Then there is a factor of $(2EV)^{-1/2}$ for each particle in the initial or final state with energy E. These factors do not depend on the nature of the interaction. The remaining part in the expression for S_{fi}, which depends on the exact nature of the interaction, is called the *Feynman amplitude*. We will denote it by $i\mathscr{M}_{fi}$. In other words, we are writing the S-matrix elements in the form:

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(\sum_i p_i - \sum_f p_f)\prod_i \frac{1}{\sqrt{2E_iV}}\prod_f \frac{1}{\sqrt{2E_fV}}\mathscr{M}_{fi}\,, \tag{6.49}$$

which serves as a definition of the Feynman amplitude. Again, p_i and p_f are the 4-momenta of initial and final particles, not components of some p.

Notice the first term on the right hand side of Eq. (6.49). This term was not present in our calculations earlier. But that is because we did not do the calculations for a case where the initial and the final state was the same, and so this term was zero for the cases we considered. But it is easy to see that this term has to be there. Consider a world where there are no interactions. In this world, the Feynman amplitude would be zero. But there should still be S-matrix elements. Since there are no interactions, the only non-zero S-matrix elements would be those corresponding to the same states initially and finally. This is the δ_{fi}-term of Eq. (6.49). Interesting

physics is of course embedded in the other term.

Let us now look at Feynman amplitudes for some of the diagrams discussed earlier. For the S-matrix element given in Eq. (6.23), the Feynman amplitude is given by

$$i\mathscr{M}_{fi}^{(1)} = (-ih)\overline{u}_s(\boldsymbol{p})v_{s'}(\boldsymbol{p}')\,, \tag{6.50}$$

where $p' = k - p$ due to the δ-function.

For the diagram in Fig. 6.2a, the contribution to the S-matrix element was given in Eq. (6.30). From this expression, we can read off the Feynman amplitude for this diagram to be:

$$\begin{aligned} i\mathscr{M}_{fi}^{(3a)} = (-ih)^3 \int \frac{d^4q}{(2\pi)^4}\, i\Delta_F(q) \\ \times \left[\overline{u}_s(\boldsymbol{p}) iS_F(p-q)\, iS_F(p-q-k) v_{s'}(\boldsymbol{p}')\right]\,. \end{aligned} \tag{6.51}$$

Similarly, for the electron-electron scattering process, the Feynman amplitude at the second order expansion in the S-matrix is given by

$$\begin{aligned} i\mathscr{M}_{fi}^{(2)} = (-ih)^2 \Big[&\overline{u}^\alpha(p_2')\overline{u}^\beta(p_1') i\Delta_F(p_1 - p_2') \\ &- \overline{u}^\alpha(p_1')\overline{u}^\beta(p_2') i\Delta_F(p_1 - p_1')\Big] u^\alpha(p_1) u^\beta(p_2), \end{aligned} \tag{6.52}$$

which can be read off from Eq. (6.48).

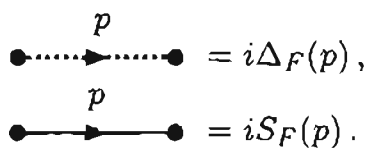
6.6 Feynman rules

The non-trivial part of writing down the S-matrix element is the determination of the Feynman amplitude. In the examples above, we have determined it by applying Wick's theorem.

However, we can draw the Feynman diagrams for a process directly by looking at the interaction Lagrangian. To do this, we first draw the initial and final particles and join them by the vertices which are allowed by the interaction Lagrangian. The number of vertices is the same as the order of perturbation theory. All vertices are labeled such that momentum and all charge conservation laws are obeyed at each vertex. Therefore, our task would be a lot easier if we had some rules to calculate $\mathscr{M}_{fi}$ directly from the Feynman diagram of a process. It turns out that a consistent set of rules, called *Feynman rules*, can be constructed from our results above. Let us try to summarize these rules here, using our experience in calculating the amplitudes.

Internal lines

Looking at the examples of the Feynman amplitudes above, it is clear how the internal lines contribute to the amplitude. If we have an internal scalar line of momentum p, there will be a factor of $i\Delta_F(p)$ in the Feynman amplitude. Similarly, if we have an internal fermion line, the appropriate factor is $iS_F(p)$. These statements are pictorially summarized as below:

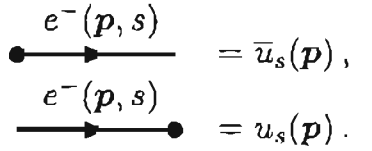


(6.53)

Of course, the fermion propagator is a matrix, so we should really put the matrix elements, as in Eq. (6.26). We will comment later on how to order the matrix indices.

External lines

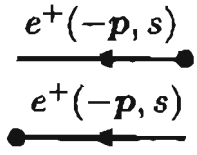
For the external lines, we can use the results in Eqs. (6.18) and (6.19). In these equations, the exponential factors on the right hand side go into constituting the momentum conserving δ-function. The factors under the square root are state factors which are outside the Feynman amplitude. The rest of the factors on the right hand side, excluding the vacuum state, go into the Feynman rule for the external particles. For scalar particles, nothing else remains after we throw out all the factors mentioned above. So we need not put anything in the Feynman rule for external scalar particles. For spin-$\frac{1}{2}$ particles, we have the following factors depending on whether it is in the initial state or in the final state:



(6.54)

Here, a small filled circle put at one end of any line implies the rest of the Feynman diagram, including all other internal and external lines. Thus the upper line shows that a particle is leaving the diagram, i.e., it is produced in the final state. In the lower diagram, the particle is annihilated.

For antiparticles in the external lines, similarly, the rules are:



(6.55)

In interpreting these rules, remember the arrow convention for antiparticles. The upper diagram now represents an antiparticle in the initial state, whereas the lower one represents an antiparticle in the final state.

Numerical factors

We have already seen that the integration over the co-ordinate of each vertex gives a factor of $(2\pi)^4$ times the momentum-conserving δ-function. If the number of vertices in a diagram be v, we will thus obtain a factor of $(2\pi)^{4v}$, and v different δ-functions.

But there are other ways in which factors of 2π would come in the amplitude. When we express the co-ordinate space propagators in terms of their Fourier transforms, it involves an integration over the momentum space, i.e., a factor of $\int d^4q/(2\pi)^4$. Thus, if there are n internal lines in a diagram, we will obtain a factor of $(2\pi)^{-4n}$. So, the overall power of 2π is $4(v-n)$. While defining the Feynman amplitude from the S-matrix element, we keep a factor of $(2\pi)^4\ \delta^4(\sum p_i - \sum p_f)$ outside. This takes care of four powers of 2π. All the other powers are left in the Feynman amplitude, which therefore has a factor of $(2\pi)^{4(v-n-1)}$. However, it can be shown that

$$v - n - 1 = -\ell\,, \tag{6.56}$$

where ℓ is the number of loops in the diagram. Thus, essentially we obtain a factor $(2\pi)^{-4\ell}$.

□ **Exercise 6.6** ⋆ Show that Eq. (6.56) holds in any connected diagram irrespective of how many lines enter any vertex.

Momentum integrations

Let us now count the number of momentum integrations. We have one momentum for each propagator from the Fourier transform, and there are n of them. But as we said before, we obtain v different δ-functions from the co-ordinate integrations at the v different vertices.

Of these, one combination can be interpreted as the momentum conservation in the overall process where no reference is made to the internal lines, as shown for example in Eq. (6.28). There will thus be $v-1$ different δ-functions which will involve the internal momenta. We can integrate over these δ-functions, which will determine some of the internal momenta in terms of the external ones. Then we will have $n-v+1$ momenta left, and using Eq. (6.56) once again, this number is seen to be the same as the number of loops, ℓ.

We can then start by writing the momenta in Feynman diagram in a way that takes care of as many δ-functions as possible. In other words, we start with the momenta of the external lines, and then start putting the momenta of the internal lines. At any stage, if the momentum of an internal line is determined by the momenta of the external lines and of the internal lines whose momenta have already been assigned, we will write it in terms of them. This way, we will have only ℓ momentum integrations in the Feynman amplitude. And since the power of 2π calculated earlier is -4ℓ, we can combine the two and say that for each loop, we need to put

$$\int \frac{d^4q}{(2\pi)^4} \tag{6.57}$$

for a momentum circulating in the loop. If we do this, we need not write down any extra numerical factor.

Dirac indices on fermion lines

In course of describing the Feynman rules so far, we have mentioned that the rules for fermions should come with appropriate Dirac indices. In case of any confusion, it is always safe to work with these indices. However, our earlier experience shows that if we follow a fermion line in the direction opposite to the arrow and put the Feynman rules for the elements encountered this way, the Dirac indices appear exactly in order, and we can do away with all the Dirac indices and treat the entire expression as a matrix formula. For fermion lines which do not close on themselves, such formulas will start with a $\overline{u}$ or a $\overline{v}$, and end with a u or v spinor. In between, there can be matrices corresponding to propagators and vertices.

For fermion loops, we can start from anywhere and follow the line in the direction opposite to the arrow. If we keep the Dirac indices,

at the end of the loop we will come back to the same index that we started with. Summing over all the Dirac indices will then imply a trace over the string of factors obtained around the loop.

Antisymmetry of fermions

The fermions anticommute among themselves. This fact manifests itself in various ways. For each fermion loop, we need to put a factor of -1, as explained at the end of §6.3.

The other consequence of this antisymmetry is a relative minus sign between diagrams differing from one another by an exchange of two identical external fermion lines. This was discussed with an example in §6.4.

Vertices

The Feynman rules for the vertices cannot be specified in such general terms, since they depend on the interaction Lagrangian which gives rise to them. Roughly speaking, if we take the term in the interaction Lagrangian, take out all the field operators, and multiply the remaining expression by a factor of i, we obtain the Feynman rule for the vertex. If there are n identical operators being taken out, we should multiply the result by a factor of $n!$. For example, for the interaction Lagrangian given in Eq. (5.9), the Feynman rule for the vertex is $-ih$. We have noticed that this is the factor coming in the S-matrix expansion for each occurrence of the vertex. On the other hand, if we had a $\lambda\phi^4$ interaction term, the Feynman rule for the corresponding vertex will be $4!i\lambda$. For other interactions, the Feynman rule can be obtained similarly. We will, of course, discuss many examples later.

6.7 Virtual particles

We observed that there may be some lines in a Feynman diagram which belong to neither the initial nor the final state. These are internal lines. There is some property of these lines which leads to an important concept in Quantum Field Theory.

To understand this with a specific example, consider now the top vertex of Fig. 6.4a. Momentum flowing through the scalar line is

$q = p_1 - p_2'$. Using $p_1^2 = p_2'^2 = m^2$ where m is the mass of the electron, we obtain $q^2 = 2p_1 \cdot q$. Suppose we use the energy-momentum relation to conclude that $q^2 = M^2$, where M is the mass of the internal boson. In a frame where the initial electron with momentum p_1 is at rest, we would then obtain $q_0 = M^2/2m$, where q_0 is the energy of the intermediate particle.

Let us stick with the case of $M > 2m$ which was used in §6.1. In that case, $q_0 > M$. On the other hand, since the energy of the initial electron in its rest frame was only m, momentum conservation cannot allow an energy larger than m for the outgoing B line.

How can both these results be correct? In quantum physics, we have to analyze carefully what can be measured. For any measurement of energy made in a finite time, there is an uncertainty. In this case, if the uncertainty is larger than the difference of the two answers obtained in two ways, we cannot really tell which answer is correct.

The internal boson was created at some time during the entire process and annihilated at some other time. If the difference between these two times is short enough, the energy uncertainty can be larger than the debated amount. Thus, this internal particle is bound to be short-lived, the extent of which is to be determined by the time-energy uncertainty relation.

We have, of course, obtained 4-momentum conservation at each vertex for any Feynman diagram, as illustrated in Ch. 6. So, for the calculation of the Feynman diagrams, we should use 4-momentum conservation, which will mean that the internal line need not satisfy the energy-momentum relation $q^2 = M^2$. This is true in general for internal lines in any Feynman diagram.

There are various words to say this. The relation $p^\mu p_\mu = m^2$ is sometimes called the on-shell condition for a particle of mass m, since it is satisfied on a sphere if we could draw the 4-dimensional graph corresponding to the components of the momentum vector. Particles on external lines, which satisfy this condition, are accordingly called *on-shell particles.* The internal particles are called *off-shell particles.* Equivalently, one uses the terms *physical particles* and *virtual particles* respectively for the same purpose. Thus, we are led to the conclusion that internal lines in a Feynman diagram can correspond to virtual particles.

This is in fact the reason that we can use propagators. Notice that the denominators of all the propagators vanish if the corresponding particle is on-shell. However, propagators are to be used for internal lines only, and if the particles in those lines are virtual, it makes the propagator meaningful. If it happens that the internal line of a Feynman diagram corresponds to a physical particle, we will have to discuss it on a different footing.

One example of this kind of diagrams is shown in Fig. 6.2b. If we cut the line going from x_2 to x_3 in this diagram, the diagram falls apart in two pieces. Such diagrams are called *1-particle reducible* (or 1PR for short) diagrams. Other examples of 1PR diagrams appear in Fig. 6.2c–d. In all of these examples, the line being cut carries the same momentum as an external physical particle. For this, we cannot use the propagator directly. The technique for calculating such diagrams will be taken up later, in Ch. 12.

Of course not all 1PR diagrams have this property. For example, the diagrams in Fig. 6.4 are also 1PR, but the internal lines represent virtual particles. So we can use the propagator to calculate the amplitude of such diagrams. In addition, we can of course discuss diagrams which do not have any such internal line which, when cut, can break the diagram into two pieces. Such diagrams are called *1-particle irreducible* (or 1PI) diagrams.

6.8 Amplitudes which are not S-matrix elements

So far, we have been talking about S-matrix elements only. These are diagrams in which all external particles are physical and observable. In practical calculations however, sometimes one has to evaluate amplitudes of diagrams where this is not the case. For example, suppose we have to evaluate the next order correction to the scattering diagrams given in Fig. 6.4. There will be no diagrams at the third order in the S-matrix expansion, but there will be a number of diagrams at the fourth order. Some examples are shown in Fig. 6.5.

There are, of course, other diagrams at this order. But it is clear that the calculation of the four displayed diagrams will be much easier if we decide to do it in two steps. In the first step, one can calculate the diagram shown in Fig. 6.6, with on-shell fermions of

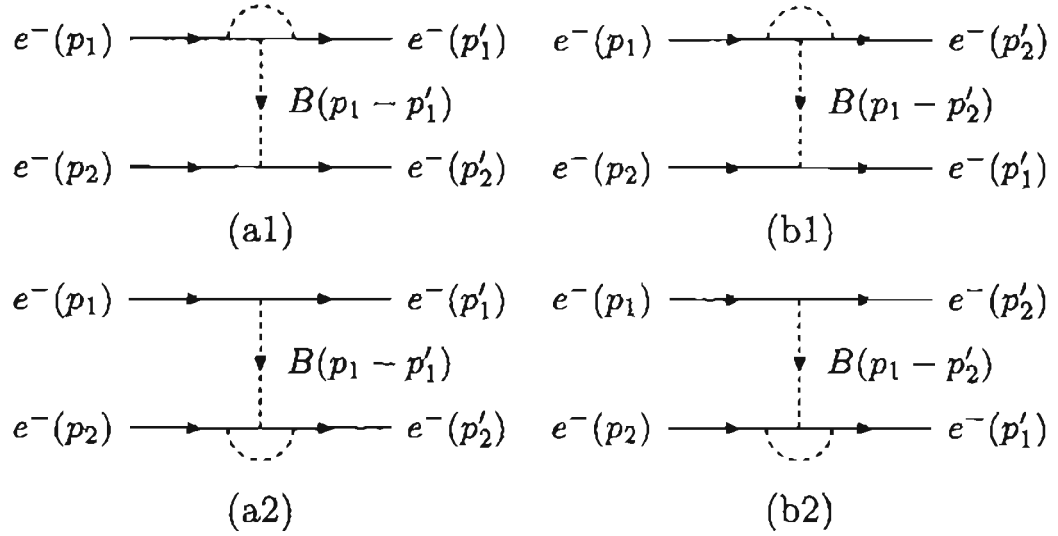


Figure 6.5: Example of diagrams whose calculation is conveniently done by evaluating the amplitude which is not an S-matrix element.

arbitrary momenta. At this stage, we do not assume anything about whether the boson line showing as an external leg of this diagram is a physical particle, and put no factor in the amplitude for this line. Once this amplitude is obtained, one can use it to find the Feynman amplitude for all the four possible diagrams of the type shown in Fig. 6.5.

Let us explain these comments in some detail. We can represent the amplitude of the diagram in Fig. 6.6 by

$$-\bar{u}(\boldsymbol{p}')V(p,p')u(\boldsymbol{p})\,, \tag{6.58}$$

where $V(p,p')$ stands for the effective vertex. Using the Feynman

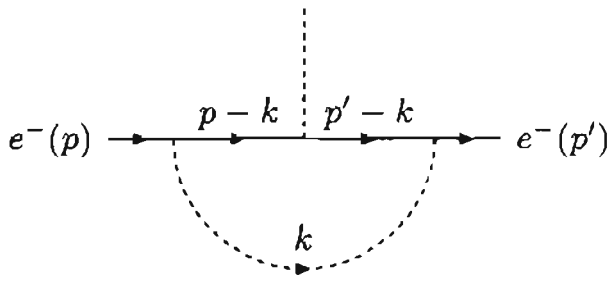


Figure 6.6: Amplitude of this diagram, calculated without assuming whether the vertical scalar line is physical or virtual, aids the calculation of the diagrams in Fig. 6.5.

rules described in §6.6, we can write the expression for this effective vertex as follows:

$$-iV(p,p') = (-ih)^3 \int \frac{d^4k}{(2\pi)^4}\, iS_F(p'-k)\, iS_F(p-k)\, i\Delta_F(k)\,, \quad (6.59)$$

where the factor $(-ih)^3$ comes from the three vertices of this diagram.

Once this is evaluated, we can write the amplitudes of the four diagrams appearing in Fig. 6.5 in the following way:

$$\begin{aligned}
i\mathscr{M}_{a1} &= [\overline{u}(p_1')\{-iV(p_1,p_1')\}u(p_1)]\, i\Delta_F(p_1-p_1')\, [\overline{u}(p_2')\{-ih\}u(p_2)] \\
i\mathscr{M}_{a2} &= [\overline{u}(p_1')\{-ih\}u(p_1)]\, i\Delta_F(p_1-p_1')\, [\overline{u}(p_2')\{-iV(p_1,p_1')\}u(p_2)] \\
i\mathscr{M}_{b1} &= [\overline{u}(p_2')\{-iV(p_1,p_2')\}u(p_1)]\, i\Delta_F(p_1-p_2')\, [\overline{u}(p_1')\{-ih\}u(p_2)] \\
i\mathscr{M}_{b2} &= [\overline{u}(p_2')\{-ih\}u(p_1)]\, i\Delta_F(p_1-p_2')\, [\overline{u}(p_1')\{-iV(p_2,p_1')\}u(p_2)],
\end{aligned} \quad (6.60)$$

where the subscripts of $\mathscr{M}$ indicate the diagram.

□ **Exercise 6.7** Find all other diagrams that will contribute to $e^- + e^+ \to e^- + e^+$ at fourth order in the S-matrix expansion.

Chapter 7

Cross sections and decay rates

The processes of highest interest are those for which the initial state has either one or two particles. In the case of one-particle initial state, we typically study its decay to a number of particles in the final state. In this case, the quantity of experimental interest is the *decay rate* of the initial particle. For two-particle initial states, we study their scattering. The final state in this case might contain the same two particles, in which case the scattering is called *elastic*. Alternatively, one can also study cases where the particle content of the initial and the final state are different, in which case the scattering is called *inelastic*. In either case, the important physical quantity is the *scattering cross section*. In this chapter, we define these quantities, along with some examples of calculations.

7.1 Decay rate

Suppose some particle decays into a number of particles in the final state. The initial and the final states are obviously not the same, so we can forget about the δ_{fi}-term. Then the S-matrix element for the process can be written as

$$S_{fi} = i(2\pi)^4 \delta^4(p_i - \sum_f p_f) \frac{1}{\sqrt{2E_i V}} \prod_f \frac{1}{\sqrt{2E_f V}} \mathscr{M}_{fi}\,, \qquad (7.1)$$

where E_i is the energy of the initial particle and E_f's are the energies of various particles in the final state. The transition probability from the initial to the final state is then given by $|S_{fi}|^2$.

However, when we try to square the matrix element we face a small problem, viz., that we need to square a δ-function. Can we assign a meaning to that? For any function $f(p)$ of momentum, we can write

$$\delta^4(p)f(p) = \delta^4(p)f(0) \tag{7.2}$$

under an integration sign. If the function $f(p)$ happens to be another δ-function, we will write

$$[\delta^4(p)]^2 = \delta^4(p)\delta^4(0)_p\,. \tag{7.3}$$

It is now necessary to assign a meaning to $\delta^4(0)_p$. This can be done if we perform our calculations within a large volume V and a large time T. Repeating the argument which led to Eq. (6.17), we can now say

$$\delta^4(0)_p = \frac{VT}{(2\pi)^4}\,. \tag{7.4}$$

Using this, we can now square the S-matrix element given in Eq. (7.1) and obtain

$$|S_{fi}|^2 = (2\pi)^4\delta^4(p_i - \sum_f p_f)VT\frac{1}{2E_iV}\prod_f\frac{1}{2E_fV}\,|\mathscr{M}_{fi}|^2\,, \tag{7.5}$$

which is the transition probability. The transition probability per unit time is obtained by dividing this expression by T, and it gives

$$|S_{fi}|^2/T = (2\pi)^4\delta^4(p_i - \sum_f p_f)\frac{1}{2E_i}\prod_f\frac{1}{2E_fV}\,|\mathscr{M}_{fi}|^2\,. \tag{7.6}$$

This is the probability per unit time of obtaining a specific final state with specific values of momenta. When we go to the infinite volume limit, the momentum values are continuous, and so we do not look for specific values of the final momenta. Rather, we ask whether the final momenta are in some specific range. If for example we are interested about whether the final momentum of some particle is in a region d^3p in the momentum space, we must multiply the above expression by the number of states in that region. This number is calculated by discretizing the phase space into cells of volume $(2\pi\hbar)^3$

and putting one state into each cell. Therefore the number of single-particle states in a momentum-space volume of d^3p is given by

$$\frac{V\,d^3p}{(2\pi)^3} \tag{7.7}$$

in natural units where $\hbar = 1$. Multiplying by this factor for all final state particles and integrating over the final momenta, we thus obtain the decay rate Γ to be

$$\Gamma = \frac{1}{2E_i}\int \prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}\,(2\pi)^4\delta^4(p_i - \sum_f p_f)\,|\mathscr{M}_{fi}|^2\,. \tag{7.8}$$

The phase space factor has to be multiplied by $1/n!$ if there are n identical particles in the final state. The lifetime of a particle is the inverse of this decay rate. In the next section, we show some examples of calculations of lifetimes with different interactions.

7.2 Examples of decay rate calculation

7.2.1 Decay of a scalar into a fermion-antifermion pair

In §6.1, we discussed the possibility of a scalar particle B decaying to e^+e^- through the Yukawa interaction term. In §6.5, we also calculated the Feynman amplitude for this process to the lowest order in the coupling constant h, which was presented in Eq. (6.50). We now use that result, and the definition of the decay rate in Eq. (7.8), to find the decay rate of the scalar B in its rest frame.

We start from the general formula of Eq. (7.8). Since in this case there are two particles in the final state, it would read

$$\Gamma = \frac{1}{2M}\int \frac{d^3p}{(2\pi)^3 2E}\int \frac{d^3p'}{(2\pi)^3 2E'}\,(2\pi)^4\delta^4(k - p - p')\,|\mathscr{M}_{fi}|^2. \tag{7.9}$$

In the factor outside the integral sign, we have replaced the energy E_i by the mass M of the decaying scalar since we are considering the decay in the rest frame of the scalar particle.

To the lowest order in the coupling constant h, the Feynman amplitude was given in Eq. (6.50). Putting that result in, we obtain

$$\Gamma = \frac{h^2}{2M}\int \frac{d^3p}{(2\pi)^3 2E}\int \frac{d^3p'}{(2\pi)^3 2E'} (2\pi)^4\delta^4(k - p - p')\,\left|\overline{u}_s(\boldsymbol{p})v_{s'}(\boldsymbol{p}')\right|^2\,. \tag{7.10}$$

This in fact will be the decay rate if we look for specific spin values s for the electron and s' for the positron. However, let us say we do not really care for the spin projections of the final particles. In that case, we should really sum over all the possible spins. Since final states with different spin projection values are incoherent, we should add the probabilities rather than the amplitudes. This gives

$$\Gamma = \frac{h^2}{2M} \int \frac{d^3p}{(2\pi)^3 2E} \int \frac{d^3p'}{(2\pi)^3 2E'} \times (2\pi)^4 \delta^4(k - p - p') \sum_{s,s'} \left|\overline{u}_s(\boldsymbol{p}) v_{s'}(\boldsymbol{p}')\right|^2 . \quad (7.11)$$

Let us now proceed to evaluate this expression, marking various stages for the sake of clarity.

Spin sum

Let us consider first the spin sum appearing in Eq. (7.11), which we denote by Σ_{spin}. Thus,

$$\begin{aligned} \Sigma_{\text{spin}} &= \sum_{s,s'} \left|\overline{u}_s(\boldsymbol{p}) v_{s'}(\boldsymbol{p}')\right|^2 \\ &= \sum_{s,s'} \left[\overline{u}_s(\boldsymbol{p}) v_{s'}(\boldsymbol{p}')\right] \left[\overline{u}_s(\boldsymbol{p}) v_{s'}(\boldsymbol{p}')\right]^* . \end{aligned} \quad (7.12)$$

Since the quantities within square brackets are numbers, we can replace the complex conjugation operation by hermitian conjugation. Thus,

$$\begin{aligned} \left[\overline{u}_s(\boldsymbol{p}) v_{s'}(\boldsymbol{p}')\right]^* &= \left[\overline{u}_s(\boldsymbol{p}) v_{s'}(\boldsymbol{p}')\right]^\dagger \\ &= \left[u_s^\dagger(\boldsymbol{p}) \gamma_0 v_{s'}(\boldsymbol{p}')\right]^\dagger \\ &= \left[v_{s'}^\dagger(\boldsymbol{p}') \gamma_0 u_s(\boldsymbol{p})\right] \\ &= \left[\overline{v}_{s'}(\boldsymbol{p}') u_s(\boldsymbol{p})\right] . \end{aligned} \quad (7.13)$$

□ **Exercise 7.1** Prove the more general case of complex conjugation when the two spinors sandwich any 4×4 matrix F:

$$\left[\overline{u}_s(\boldsymbol{p}) F v_{s'}(\boldsymbol{p}')\right]^* = \left[\overline{v}_{s'}(\boldsymbol{p}') F^\ddagger u_s(\boldsymbol{p})\right] , \quad (7.14)$$

where

$$F^\ddagger \equiv \gamma_0 F^\dagger \gamma_0 . \quad (7.15)$$

Let us now go back to Eq. (7.12) and rewrite it using Eq. (7.13) and putting the spinor indices explicitly:

$$\Sigma_{\text{spin}} = \sum_{s,s'} \left[(\overline{u}_s(\boldsymbol{p}))_\alpha \, (v_{s'}(\boldsymbol{p}'))_\alpha \right] \left[(\overline{v}_{s'}(\boldsymbol{p}'))_\beta \, (u_s(\boldsymbol{p}))_\beta \right]$$
$$= \left[\sum_s u_s(\boldsymbol{p}) \overline{u}_s(\boldsymbol{p}) \right]_{\beta\alpha} \left[\sum_{s'} v_{s'}(\boldsymbol{p}') \overline{v}_{s'}(\boldsymbol{p}') \right]_{\alpha\beta} . \qquad (7.16)$$

Notice that we have now separated the sums over s and s', and these sums over single spin variables were given in Eq. (4.53). Using the results, we can now write

$$\Sigma_{\text{spin}} = [\not{p} + m]_{\beta\alpha} \, [\not{p}' - m]_{\alpha\beta}$$
$$= \text{Tr} \, [(\not{p} + m) \, (\not{p}' - m)] \, . \qquad (7.17)$$

□ **Exercise 7.2** For the more general case when the matrix element is of the form $\overline{u}_s(\boldsymbol{p}) F v_{s'}(\boldsymbol{p}')$ with any 4×4 matrix F, show that the spin sum of the absolute square of the matrix element can be written as

$$\sum_{s,s'} |\overline{u}_s(\boldsymbol{p}) F v_{s'}(\boldsymbol{p}')|^2 = \text{Tr} \, [(\not{p} + m) \, F \, (\not{p}' - m) F^\dagger] \, , \qquad (7.18)$$

where $F^\dagger$ was defined in Eq. (7.15).

Traces of γ-matrices

We now have to evaluate the trace appearing in Eq. (7.17). We can decompose it into four terms:

$$\text{Tr} \, [(\not{p} + m) \, (\not{p}' - m)] = \text{Tr} \, (\not{p}\not{p}' - m\not{p} + m\not{p}' - m^2) \, . \qquad (7.19)$$

Here, m is a number which can be taken outside of the trace operation. The last term is then really m^2 times the trace of the unit matrix, which gives $4m^2$. The second term can be written as

$$m p^\mu \, \text{Tr} \, (\gamma_\mu) = 0 \, , \qquad (7.20)$$

since the trace of any γ-matrix vanishes, as was proved in Eq. (4.10). For the same reason, the third term also vanishes. In fact, one can prove a more general result, viz., that the trace of any odd number of γ-matrices vanishes. This and some other useful trace formulas are given in Appendix A.2.

We are now left with the first trace appearing in Eq. (7.19). This can be rewritten as

$$p^\mu p'^\nu \mathrm{Tr}\,(\gamma_\mu \gamma_\nu)\,. \tag{7.21}$$

However, since the trace operation is cyclic, we can also write it as

$$p'^\mu p^\nu \mathrm{Tr}\,(\gamma_\nu \gamma_\mu)\,, \tag{7.22}$$

or equivalently, as

$$\frac{1}{2} p^\mu p'^\nu \mathrm{Tr}\,(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = p^\mu p'^\nu \mathrm{Tr}\,(g_{\mu\nu})\,, \tag{7.23}$$

using the anticommutation relation of the γ-matrices in the last step. Now, $g_{\mu\nu}$ is just a number so far as the spinorial indices are concerned, so the trace is really $g_{\mu\nu}$ times the trace of the unit matrix. Thus Eq. (7.23) gives us $4p \cdot p'$. Putting it in Eq. (7.19), we obtain

$$\mathrm{Tr}\,[(\not{p} + m)\,(\not{p} - m)] = 4(p \cdot p' - m^2). \tag{7.24}$$

Inserting the expression for the spin sum in Eq. (7.11), we can write

$$\Gamma = \frac{h^2}{2M} \int \frac{d^3p}{(2\pi)^3 2E} \int \frac{d^3p'}{(2\pi)^3 2E'} (2\pi)^4 \delta^4(k - p - p')\, 4(p \cdot p' - m^2)\,. \tag{7.25}$$

First we evaluate the final factor, which came from the spin sum, subject to the constraints of the momentum-conserving δ-function. The δ-function enforces $k = p + p'$, and squaring both sides, we get $2p \cdot p' = M^2 - 2m^2$. Therefore,

$$4(p \cdot p' - m^2) = 2(M^2 - 4m^2) \tag{7.26}$$

when multiplied by the δ-function.

Phase space

Thus the factor coming from the Feynman amplitude becomes independent of the magnitude or direction of the 3-momenta, and can be pulled out of the integral in Eq. (7.25). So we can write

$$\Gamma = \frac{h^2\,(M^2 - 4m^2)}{M}\,\rho\,, \tag{7.27}$$

where

$$\rho = \int \frac{d^3p}{(2\pi)^3 2E} \int \frac{d^3p'}{(2\pi)^3 2E'} (2\pi)^4 \delta^4(k - p - p') . \qquad (7.28)$$

This ρ is usually called the *phase space factor*. It contains all kinematical factors, i.e., factors not involving the Feynman amplitude and the initial state factor. It also includes the δ-function for the 4-momentum conservation. Of course, it is not always possible to make such a separation into a phase space factor and the Feynman amplitude factor. It works here and in other problems where the factors other than ρ do not depend on the magnitude or direction of the 3-momenta. Later, we will also encounter examples where this separation cannot be done. However, in this case, the separation works, and it is very convenient to do things piecewise. Let us now evaluate the phase space factor.

Starting from Eq. (7.28), we can immediately integrate over one of the final momenta, say $\boldsymbol{p}'$. This gives

$$\rho = \frac{1}{(2\pi)^2} \int \frac{d^3p}{4EE'} \delta(k_0 - p_0 - p'_0) . \qquad (7.29)$$

Since we have decided to perform the calculations in the rest frame of the decaying particle,

$$k^\mu = (M, 0, 0, 0) . \qquad (7.30)$$

In this frame, let us denote the 0-components of p and p' by E and E' respectively. Since the mass of the fermion is m, we must have $E^2 - \mathbf{p}^2 = m^2$ as well as $E'^2 - \mathbf{p}'^2 = m^2$. But we have already used the spatial δ-functions, so $\boldsymbol{p}' = -\boldsymbol{p}$ in this frame. It follows that $E = E'$, so that

$$\begin{aligned} \rho &= \frac{1}{(2\pi)^2} \int \frac{d^3p}{4E^2} \delta(M - 2E) \\ &= \frac{1}{\pi} \int \frac{d\mathbf{p}\ \mathbf{p}^2}{4E^2} \cdot \frac{1}{2} \delta(E - \frac{1}{2}M) . \end{aligned} \qquad (7.31)$$

In the last step, we have integrated over the angular variables of the 3-vector $\boldsymbol{p}$, noting the fact that the integrand does not depend on them. This angular integration gives a factor 4π.

Because of the energy-momentum relation, we can write

$$d\mathbf{p}\ \mathbf{p} = dE\ E\,. \tag{7.32}$$

The δ-function forces E to take the value $M/2$, and therefore $\mathbf{p} = \sqrt{E^2 - m^2} = \frac{1}{2}M\sqrt{1-(4m^2/M^2)}$. From Eq. (7.31), we can now write

$$\begin{aligned}\rho &= \frac{1}{8\pi}\int \frac{dE\,\mathbf{p}}{E}\cdot\delta(E-\tfrac{1}{2}M) \\ &= \frac{1}{8\pi}\sqrt{1-\frac{4m^2}{M^2}}\,.\end{aligned} \tag{7.33}$$

Putting everything together, we find from Eq. (7.27) that

$$\Gamma = \frac{Mh^2}{8\pi}\left(1-\frac{4m^2}{M^2}\right)^{3/2}\,. \tag{7.34}$$

The lifetime is the inverse of Γ, and therefore is inversely proportional to h^2. In other words, if h is smaller, the lifetime will be longer. This is expected intuitively — if the coupling is smaller, it will take more time for the process to take place. Secondly, for Γ to be real, one must have $M > 2m$. This is also expected, since if $M < 2m$, the decay cannot take place because of energy conservation.

□ **Exercise 7.3** ⋆ Calculate the decay rate for the process $B \to e^-e^+$ in a frame where the initial B particle moves with a speed v in the z-direction, i.e., its 4-momentum is given by $k^\mu = \gamma M(1,0,0,v)$ where $\gamma = (1-v^2)^{-1/2}$. You may neglect the mass of the electron for this calculation. Is your final result justifiable by a time-dilation argument?

□ **Exercise 7.4** Suppose the interaction between the spinless field and the fermions is given by Eq. (5.10), instead of Eq. (5.9) which was used in the calculation above. Show that in this case, one obtains the same formula for the decay rate, i.e., Eq. (7.34), with h substituted by $|h'|$.

7.2.2 Muon decay with 4-fermion interaction

We now calculate the decay rate of an unstable fermion, the muon. It decays predominantly via the process

$$\mu^-(p) \to e^-(p') + \widehat{\nu}_e(k') + \nu_\mu(k)\,, \tag{7.35}$$

where ν_μ is called the *muon-neutrino*, ν_e is the *electron-neutrino* and the hat indicates the antiparticle. The 4-momenta of the particles involved are given in parentheses.

The interaction Lagrangian governing the decay of the muon is of the 4-fermion type, which can be written as

$$\mathscr{L}_{\rm int} = \frac{G_F}{\sqrt{2}}\,\overline{\psi}_{(e)}\gamma^\lambda(1-\gamma_5)\psi_{(\nu_e)}\,\overline{\psi}_{(\nu_\mu)}\gamma_\lambda(1-\gamma_5)\psi_{(\mu)}\,. \qquad (7.36)$$

Here G_F is called the Fermi constant, whose value is $1.166 \times 10^{-5}\,\mathrm{GeV}^{-2}$. Of course, the hermitian conjugate term must also be present in order that the Lagrangian is hermitian, but we will not need that term for describing this decay. The interaction Hamiltonian arising from this term is simply

$$\mathscr{H}_I = -\frac{G_F}{\sqrt{2}}\,\overline{\psi}_{(e)}\gamma^\lambda(1-\gamma_5)\psi_{(\nu_e)}\,\overline{\psi}_{(\nu_\mu)}\gamma_\lambda(1-\gamma_5)\psi_{(\mu)}\,. \qquad (7.37)$$

The lowest order Feynman amplitude for the process is then given by

$$\mathscr{M}_{fi} = \frac{G_F}{\sqrt{2}}\,\overline{u}_{(e)}(p')\gamma^\lambda(1-\gamma_5)v_{(\nu_e)}(k')\,\overline{u}_{(\nu_\mu)}(k)\gamma_\lambda(1-\gamma_5)u_{(\mu)}(p)\,. \qquad (7.38)$$

Let us find the decay rate of unpolarized muons. For this we sum over initial muon spin and divide by 2. Summing over the final spins, we get the decay rate of the muon,

$$\Gamma = \frac{1}{2m_\mu}\int\frac{d^3p'}{(2\pi)^3 2p'_0}\int\frac{d^3k}{(2\pi)^3 2k_0}\int\frac{d^3k'}{(2\pi)^3 2k'_0} \times(2\pi)^4\delta^4(p-p'-k-k')\times\frac{1}{2}\sum_{\rm spin}|\mathscr{M}_{fi}|^2\,. \qquad (7.39)$$

Here m_μ is the mass of the muon, which appears in the denominator since we are calculating the decay rate in the rest frame of the muon. Since the subscript μ here stands for the muon, we will not use the same subscript for Lorentz indices in this section, in order to avoid any possible confusion.

Spin sum

The first task, as before, is to calculate the square of the Feynman amplitude. From the expression of the amplitude in Eq. (7.38), we can write

$$\begin{aligned}\frac{1}{2}\sum_{\text{spin}}|\mathscr{M}_{fi}|^2 = \frac{G_F^2}{4}\sum_{\text{spin}} & \left[\overline{u}_{(e)}(p')\gamma^\lambda(1-\gamma_5)v_{(\nu_e)}(k')\right] \\ & \times \left[\overline{u}_{(\nu_\mu)}(k)\gamma_\lambda(1-\gamma_5)u_{(\mu)}(p)\right] \\ & \times \left[\overline{u}_{(e)}(p')\gamma^\rho(1-\gamma_5)v_{(\nu_e)}(k')\right]^\dagger \\ & \times \left[\overline{u}_{(\nu_\mu)}(k)\gamma_\rho(1-\gamma_5)u_{(\mu)}(p)\right]^\dagger . \quad (7.40)\end{aligned}$$

We shall take the neutrinos to be massless, because current experimental data shows that even if the neutrinos are massive, they are much lighter than all charged particles. Then the spin sum rules described in §7.2.1 yield

$$\begin{aligned}\overline{|\mathscr{M}|^2} = \frac{G_F^2}{4}\,\mathrm{Tr}\left[(\not{p}'+m_e)\gamma^\lambda(1-\gamma_5)\not{k}'\gamma^\rho(1-\gamma_5)\right] \\ \times\,\mathrm{Tr}\left[\not{k}\gamma_\lambda(1-\gamma_5)(\not{p}+m_\mu)\gamma_\rho(1-\gamma_5)\right] . \quad (7.41)\end{aligned}$$

From now on, we will write $\overline{|\mathscr{M}|^2}$ for the absolute square of the Feynman amplitude, suitably summed or averaged over various spins as the problem demands. Now, γ_5 anticommutes with any of the γ_λ, and so it commutes with a string of an even number of γ-matrices. Using this, we can write

$$\begin{aligned}(1-\gamma_5)\not{k}'\gamma^\rho(1-\gamma_5) &= \not{k}'\gamma^\rho(1-\gamma_5)^2 \\ &= 2\not{k}'\gamma^\rho(1-\gamma_5) , \quad (7.42)\end{aligned}$$

using the property $(\gamma_5)^2 = 1$ in the last step. Then we can write

$$\begin{aligned}\overline{|\mathscr{M}|^2} = G_F^2\,\mathrm{Tr}\left[(\not{p}'+m_e)\gamma^\lambda\not{k}'\gamma^\rho(1-\gamma_5)\right] \\ \times\,\mathrm{Tr}\left[\not{k}\gamma_\lambda(\not{p}+m_\mu)\gamma_\rho(1-\gamma_5)\right] . \quad (7.43)\end{aligned}$$

In either trace, the term proportional to the mass vanishes, since it involves an odd number of γ-matrices. For the remaining terms, we can use the trace formulas given in Appendix A.2 to write

$$\begin{aligned}\overline{|\mathscr{M}|^2} = 16G_F^2\left[p'^\lambda k'^\rho + p'^\rho k'^\lambda - g^{\lambda\rho}p'\cdot k' - i\varepsilon_{\lambda\sigma\rho\tau}p'^\sigma k'^\tau\right] \\ \times\left[p_\lambda k_\rho + p_\rho k_\lambda - g_{\lambda\rho}p\cdot k - i\varepsilon^{\lambda\nu\rho\eta}k_\nu p_\eta\right] . \quad (7.44)\end{aligned}$$

Notice that in each trace factor, the first three terms are symmetric in the Lorentz indices λ, ρ, whereas the last term is antisymmetric. Thus, the contraction of the first three terms with the last one of the other trace vanishes, and we are left with

$$\overline{|\mathscr{M}|^2} = 16G_F^2\Big[\Big(p'^\lambda k'^\rho + p'^\rho k'^\lambda - g^{\lambda\rho}p'\cdot k'\Big)\Big(p_\lambda k_\rho + p_\rho k_\lambda - g_{\lambda\rho}p\cdot k\Big) - \varepsilon_{\lambda\sigma\rho\tau}\varepsilon^{\lambda\nu\rho\eta}p'^\sigma k'^\tau k_\nu p_\eta\Big]. \tag{7.45}$$

The result of the contraction among the symmetric parts of the two traces is given by

$$2\,(p\cdot p'\; k\cdot k' + p\cdot k'\; k\cdot p')\;. \tag{7.46}$$

As for the other parts involving the ε-tensors, we use the identity of Eq. (A.33),

$$\varepsilon_{\lambda\sigma\rho\tau}\varepsilon^{\lambda\nu\rho\eta} = -2\,(g_\sigma{}^\nu\; g_\tau{}^\eta - g_\tau{}^\nu\; g_\sigma{}^\eta)\;, \tag{7.47}$$

which gives for the last contraction the result

$$\varepsilon_{\lambda\sigma\rho\tau}\varepsilon^{\lambda\nu\rho\eta}p'^\sigma k'^\tau k_\nu p_\eta = -2\,(p\cdot k'\; k\cdot p' - p\cdot p'\; k\cdot k')\;. \tag{7.48}$$

Combining the two terms and noting the negative sign between the two in Eq. (7.45), we obtain the final result for the square of the Feynman amplitude in the following form:

$$\overline{|\mathscr{M}|^2} = \frac{1}{2}\sum_{\text{spin}}|\mathscr{M}_{fi}|^2 = 64G_F^2\; p\cdot k'\; k\cdot p'\,. \tag{7.49}$$

Inserting this expression into Eq. (7.39), we can write

$$\Gamma = \frac{G_F^2}{\pi^5 m_\mu}\int\frac{d^3p'}{2p'_0}\int\frac{d^3k}{2k_0}\int\frac{d^3k'}{2k'_0}\;\delta^4(p-p'-k-k')\;p\cdot k'\;k\cdot p'\,. \tag{7.50}$$

Integration over k and k'

The integration is substantially more complicated than in the example shown in §7.2.1 because the final state here has three particles. So, we perform the integration in steps. First, we write Eq. (7.50) as

$$\Gamma = \frac{G_F^2}{\pi^5 m_\mu}\int\frac{d^3p'}{2p'_0}\;p^\lambda p'^\rho I_{\lambda\rho}(p-p')\,, \tag{7.51}$$

where, writing $q = p - p'$, we can dump all the terms depending on k and k' into the definition of $I_{\lambda\rho}$:

$$I_{\lambda\rho}(q) = \int \frac{d^3k}{2k_0} \int \frac{d^3k'}{2k'_0}\, \delta^4(q - k - k') k'_\lambda k_\rho \,. \tag{7.52}$$

Note that the integration measures over the final state momenta can be written as

$$\int \frac{d^3p}{2E} = \int d^4p\, \delta(p^2 - m^2)\,, \tag{7.53}$$

where m is the mass of the relevant particle. The form on the right hand side makes it obvious that the integration measure is a Lorentz invariant quantity. Thus, the entire thing under the integration sign in Eq. (7.51) is Lorentz invariant.

Therefore $I_{\lambda\rho}$ must be a rank-2 tensor, and can depend only on the 4-vector q. So the most general form for $I_{\lambda\rho}$ can be written as

$$I_{\lambda\rho} = Aq^2 g_{\lambda\rho} + Bq_\lambda q_\rho\,, \tag{7.54}$$

where A and B are Lorentz invariants which need to be determined. From dimensional analysis, we see that both A and B have to be dimensionless.

We have now two different expressions for $I_{\lambda\rho}$, which are given in Eqs. (7.52) and (7.54). Contracting both these expressions by $g^{\lambda\rho}$ and equating the results, we obtain

$$(4A + B)q^2 = \int \frac{d^3k}{2k_0} \int \frac{d^3k'}{2k'_0}\, \delta^4(q - k - k')\; k' \cdot k\,. \tag{7.55}$$

Similarly, contracting the two equations by $q^\lambda q^\rho$, we get

$$\begin{aligned}(A + B)q^4 &= \int \frac{d^3k}{2k_0} \int \frac{d^3k'}{2k'_0}\, \delta^4(q - k - k')\; q \cdot k'\; q \cdot k \\ &= \int \frac{d^3k}{2k_0} \int \frac{d^3k'}{2k'_0}\, \delta^4(q - k - k')\; (k' \cdot k)^2\,,\end{aligned} \tag{7.56}$$

where in the last step, we have replaced q by $k + k'$ outside the δ-function, and put $k^2 = k'^2 = 0$ for the neutrinos.

Eqs. (7.55) and (7.56) have Lorentz invariant quantities on both sides, so we can choose any frame for evaluating these integrals. A

convenient frame is one in which $\boldsymbol{k} + \boldsymbol{k}' = 0$, i.e., the neutrinos move with equal and opposite 3-momenta. Since both the neutrinos are massless, this also implies $k_0 = k_0' = \mathrm{k}$. And therefore, denoting the results of this frame by a subscript 'CM', we obtain

$$(k \cdot k')_{\mathrm{CM}} = k_0 k_0' - \boldsymbol{k} \cdot \boldsymbol{k}' = k_0 k_0' + \mathrm{k}^2 = 2k_0^2 \,. \tag{7.57}$$

Inserting this into Eq. (7.55) and performing the k' integration, we obtain

$$\left((4A + B)q^2\right)_{\mathrm{CM}} = \int \frac{d^3k}{2k_0} \frac{1}{2k_0} \, \delta(q_0 - 2k_0) \, 2k_0^2 \,. \tag{7.58}$$

The angular integrations give a factor of 4π, and the integration over the magnitude of $\boldsymbol{k}$ is easily performed by using the δ-function. This gives $\pi q_0^2/4$ for the right hand side. This result is obtained in the CM frame where $\boldsymbol{q} = 0$, i.e., $q^2 = q_0^2$. So

$$4A + B = \frac{\pi}{4} \,. \tag{7.59}$$

Similarly, performing the integration of Eq. (7.56) in the same frame, we obtain

$$A + B = \frac{\pi}{8} \,. \tag{7.60}$$

Solving these two equations, we find A and B. Putting it in Eq. (7.54), we get

$$I_{\lambda\rho} = \frac{\pi}{24} \left(q^2 g_{\lambda\rho} + 2q_\lambda q_\rho\right) \,. \tag{7.61}$$

Putting this back into Eq. (7.51), we can write

$$\begin{aligned} \Gamma &= \frac{G_F^2}{24\pi^4 m_\mu} \int \frac{d^3p'}{2p_0'} \, p^\lambda p'^\rho \left(q^2 g_{\lambda\rho} + 2q_\lambda q_\rho\right) \\ &= \frac{G_F^2}{24\pi^4 m_\mu} \int \frac{d^3p'}{2p_0'} \left(q^2 p \cdot p' + 2p \cdot q p' \cdot q\right) \,, \end{aligned} \tag{7.62}$$

where $q = p - p'$ by definition.

Differential decay rate

Before performing the remaining integrations, let us observe a few things. We are calculating the decay rate in the rest frame of the muon, with

$$p^\lambda = (m_\mu, 0, 0, 0)\,. \tag{7.63}$$

In this frame, let us write the electron momentum as

$$p'^\lambda = (E_e, \boldsymbol{p}')\,. \tag{7.64}$$

This means

$$p \cdot p' = m_\mu E_e \tag{7.65}$$

in this frame. Moreover,

$$q^2 = (p - p')^2 = m_\mu^2 + m_e^2 - 2m_\mu E_e\,, \tag{7.66}$$

$$p \cdot q = p^2 - p \cdot p' = m_\mu^2 - m_\mu E_e\,, \tag{7.67}$$

$$p' \cdot q = p \cdot q - q^2 = m_\mu E_e - m_e^2\,. \tag{7.68}$$

Plugging all this into Eq. (7.62), we get

$$\Gamma = \frac{G_F^2}{24\pi^4} \int \frac{d^3p'}{2E_e} \left[\left(m_\mu^2 + m_e^2 - 2m_\mu E_e\right) E_e + 2\left(m_\mu - E_e\right)\left(m_\mu E_e - m_e^2\right) \right]. \tag{7.69}$$

In the integrand, nothing depends on the angular variables, so they can be integrated trivially. Using Eq. (7.32) on the electron momentum, we can then write

$$\Gamma = \frac{G_F^2}{12\pi^3} \int dE_e\, \mathbf{p}' \left[\left(m_\mu^2 + m_e^2 - 2m_\mu E_e\right) E_e + 2\left(m_\mu - E_e\right)\left(m_\mu E_e - m_e^2\right) \right]. \tag{7.70}$$

We can write this as

$$\Gamma = \int dE_e\, \frac{d\Gamma}{dE_e}\,, \tag{7.71}$$

where

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2}{12\pi^3}\sqrt{E_e^2 - m_e^2}\ \left[\left(m_\mu^2 + m_e^2 - 2m_\mu E_e\right) E_e \right. \\ \left. +2\left(m_\mu - E_e\right)\left(m_\mu E_e - m_e^2\right)\right]. \quad (7.72)$$

This quantity then represents the transition rate per unit energy into final states with electron energy between E_e and $E_e + dE_e$. In other words, if we observe electrons coming out of the decay of a large number of muons, the probability that the electron energy is between E_e and $E_e + dE_e$ is given by

$$\frac{1}{\Gamma}\frac{d\Gamma}{dE_e}\, dE_e\,. \quad (7.73)$$

Often, this kind of distributions are very useful in checking how theoretical predictions agree with experimental results. Since it involves the derivative of the decay rate with respect to some final state parameter, such quantities are called *differential decay rates*. In this case, the derivative is with respect to the electron energy, but we could have considered other kinematical variables as well. For example, we could take the derivative with respect to the energy of the ν_μ. But experimentally, that quantity would not have been of any interest since the neutrino energies are not directly measured in an experiment. Another possible variable would be the direction of the electron momentum, but in this case $\overline{|\mathscr{M}|^2}$ does not depend on this direction. So the derivative would be uninteresting.

Total decay rate

The total decay rate can be obtained by integrating Eq. (7.72) with respect to E_e. In order to perform this integration, we will make one simplifying assumption, viz., that the mass of the electron is zero. This is a good assumption, since $m_\mu = 106\,\text{MeV}$, whereas $m_e = 0.511\,\text{MeV}$, so that the electron mass is negligible compared to the muon mass. With this assumption, Eq. (7.72) can be written as

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2 m_\mu}{12\pi^3} E_e^2\left(3m_\mu - 4E_e\right)\,. \quad (7.74)$$

To integrate this, we need the limits of the integration. The lower limit of the electron energy can certainly be zero if the neutrinos carry

the entire energy. The upper limit is obtained when the two neutrinos are emitted in the same direction, opposite to the direction of the electron. In this case, $\mathbf{p}' = \mathbf{k} + \mathbf{k}'$, which also means $E_e = k_0 + k'_0$ since all the decay products are now considered massless. However, $E_e + k_0 + k'_0 = m_\mu$ due to energy conservation. So we obtain that in this case, the energy of the electron is $m_\mu/2$, which should be the upper limit of integration. Therefore, the total decay rate is given by

$$\begin{aligned}\Gamma &= \frac{G_F^2 m_\mu}{12\pi^3}\int_0^{m_\mu/2} dE_e\, E_e^2\,(3m_\mu - 4E_e) \\ &= \frac{G_F^2 m_\mu^5}{192\pi^3}\,. \end{aligned} \tag{7.75}$$

This lifetime was quoted earlier in Eq. (1.43).

□ **Exercise 7.5** Start from the expression for the differential decay rate in Eq. (7.72). Do not completely neglect the electron mass. Find the corrections to the total decay rate at the lowest order of m_e/m_μ. Using $m_e = 0.511\,\mathrm{MeV}$ and $m_\mu = 106\,\mathrm{MeV}$, estimate the fractional error in assuming $m_e = 0$.

□ **Exercise 7.6** Two of the possible decay modes, or channels, of the charged pion π^+ are $\pi^+ \to \mu^+ + \nu_\mu$ and $\pi^+ \to e^+ + \nu_e$. In either case, the interaction Lagrangian is

$$\mathscr{L}_{\rm int} = G_F f_\pi (\partial_\lambda \phi)\, \overline{\psi}_\ell\, \gamma^\lambda (1-\gamma_5)\psi_\nu + \text{h.c.}\,, \tag{7.76}$$

where f_π is a constant (called the pion decay constant) which is the same for both processes, ψ_ℓ is the field for the charged lepton (electron or muon) and ψ_ν is the field for the corresponding neutrino. Find the ratio of the decay rates in these two channels. [Hint: Derivative interactions are treated like any other interaction in perturbation theory, ∂_λ giving a factor of k_λ in the Feynman rule.]

7.3 Scattering cross section

At the beginning of this chapter, we mentioned that the concept of scattering cross section is important when the initial state in the S-matrix contains two particles. We now quantify the concept.

The name "cross-section" derives from a paradigm scattering process involving hard spheres. Consider, for example, a hard sphere of radius a, located somewhere within a total area of A. Another sphere is thrown towards it. To this sphere, the target sphere shows an area of πa^2. Thus the probability that the incoming sphere would scatter

from this target sphere is given by $\pi a^2/A$. Calling this the probability of a scattering event and denoting it by P_S, we can then write

$$\pi a^2 = P_S A\,. \tag{7.77}$$

This equation gives the cross sectional area of the sphere in terms of the beam area.

All objects are of course not hard spheres. But in a scattering process, we can still define the effective cross section by the same formula. The cross section is usually denoted by the symbol σ. Thus, by definition,

$$\sigma = P_S A\,. \tag{7.78}$$

Now suppose we have a parallel beam with a density ρ and velocity v towards the target. In time t, this beam fills a volume $\rho v t A$, where A is now the area normal to the beam which fully contains it. Choosing t such that the volume contains just one particle, we can write

$$1 = \rho v t A\,, \tag{7.79}$$

or,

$$A = \frac{1}{\rho v t}\,. \tag{7.80}$$

Therefore, Eq. (7.78) can be rewritten as

$$\sigma = \frac{P_S/t}{\rho v}\,. \tag{7.81}$$

The quantity P_S/t appearing in the numerator is called the transition rate, i.e., the probability of scattering per unit time. The quantity in the denominator is the flux of particles. Thus the cross section can be defined as the transition rate per unit of incident flux.

Although we have written these formulas for classical particles, we will now carry everything over to quantum mechanical scattering. P_S will now mean the quantum mechanical transition probability. Consider a scattering process which takes an initial state $|i\rangle$ at $t \to -\infty$ to a final state $|f\rangle$ at $t \to +\infty$. The S-matrix element between

these two states is given by

$$\begin{aligned} S_{fi} &= \langle f\,|S|\,i\rangle \\ &= \delta_{fi} + i(2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right)\prod_i \frac{1}{\sqrt{2E_iV}}\prod_f \frac{1}{\sqrt{2E_fV}}\mathscr{M}_{fi}\,. \end{aligned} \tag{7.82}$$

The δ_{fi}-term stands for the case when nothing happened, and is of no interest to us. Throwing it out and squaring the matrix element, we obtain

$$\left|S_{fi}\right|^2 = (2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right)VT\prod_i \frac{1}{2E_iV}\prod_f \frac{1}{2E_fV}\left|\mathscr{M}_{fi}\right|^2\,, \tag{7.83}$$

where the square of the momentum-conserving δ-function has given us VT, according to §7.1. As in there, we have assumed that the scattering process occurs within a large volume V and a large time T, both of which will be taken to infinity at the end.

This gives the transition probability. The transition rate, or the transition probability per unit time, is therefore given by

$$\frac{\left|S_{fi}\right|^2}{T} = (2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right)V\prod_i \frac{1}{2E_iV}\prod_f \frac{1}{2E_fV}\left|\mathscr{M}_{fi}\right|^2\,, \tag{7.84}$$

This is the transition rate to a final state of specified momentum. For a range of momenta, we should integrate this, including a factor of $V\,d^3p/(2\pi)^3$ for the number of states. This gives

$$\sigma = \frac{1}{\rho v V}\frac{1}{4E_1E_2}\int\left(\prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}\right)(2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right)\left|\mathscr{M}_{fi}\right|^2, \tag{7.85}$$

where we have written the initial state factors explicitly.

As for the flux, we have taken the incident state normalized to one particle in the entire volume, i.e., the density of particles in the initial state to be $1/V$. When we considered one beam particle

hitting one target particle at rest, we had to multiply the density by the velocity of the beam particle in order to obtain the flux. In general, both particles may be moving, and for v we should use

$$v_{\text{rel}} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}\,, \tag{7.86}$$

where $\boldsymbol{p}_i$ and E_i denote, for $i = 1, 2$, the 3-momenta and the energies of the initial particles whose masses are m_1 and m_2. In a collinear frame, i.e., if $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are along the same line, this reduces to

$$v_{\text{rel}} = \left| \frac{\boldsymbol{p}_1}{E_1} - \frac{\boldsymbol{p}_2}{E_2} \right| . \tag{7.87}$$

This coincides with relative velocity for small velocities, but not in general. Then we can write the definition of the scattering cross section as

$$\sigma = \frac{1}{v_{\text{rel}}} \frac{1}{4E_1 E_2} \int \prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f} (2\pi)^4 \delta^4 \left(\sum_i p_i - \sum_f p_f \right) |\mathscr{M}_{fi}|^2 . \tag{7.88}$$

7.4 Generalities of 2-to-2 scattering

A scattering process has two particles in the initial state. But there can be any number of particles in the final state. However, the calculations of cross sections become particularly simple if the final state also contains two particles. In this case, it is also easier to analyze experimental data. Such processes are called 2-to-2 scattering processes. In this section, we show some generalities of the calculation of scattering cross-section in such cases.

Let us symbolically denote the scattering process in the following way:

$$1(p_1, m_1) + 2(p_2, m_2) \to 1'(p_1', m_1') + 2'(p_2', m_2')\,, \tag{7.89}$$

where p_1 and p_2 are the 4-momenta for the initial particles 1 and 2, which have masses m_1 and m_2. For the final particles $1'$ and $2'$, we denote the momenta and masses with a prime.

From the general formula given in Eq. (7.88), we can write down the expression for the cross-section for the present case:

$$\sigma = \frac{1}{v_{\rm rel}} \frac{1}{4E_1E_2} \int \frac{d^3p_1'}{(2\pi)^3 2E_1'} \int \frac{d^3p_2'}{(2\pi)^3 2E_2'} (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \, \overline{|\mathscr{M}|^2} \,. \quad (7.90)$$

Of course, $\overline{|\mathscr{M}|^2}$ depends on the process we are considering and the interaction Hamiltonian we have for that. Without a detailed knowledge of that, we cannot find out the cross section. However, already the kinematics can give us some simplification in the formula.

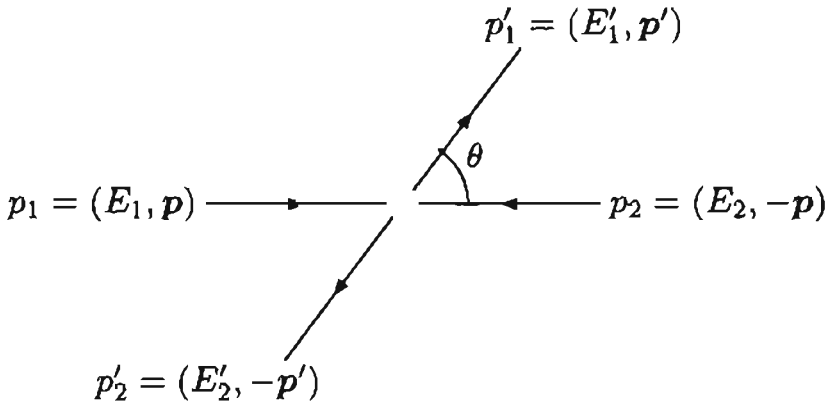


Figure 7.1: Kinematics of two particle scattering in the center-of-mass frame.

The analysis is usually done in two types of frames:

1. The Center-of-Mass (or CM) frame, which is defined by the condition that the total 3-momentum of the initial particles is zero in this frame.

2. The Lab frame, in which one of the initial particles is at rest.

We should mention that the phrase "Lab frame" is somewhat confusing. These days many experiments are performed in which none of the initial particles is at rest. To make matters more confusing, in many modern experiments the physical frame of the laboratory is really the center-of-mass frame. Nevertheless, the name "Lab frame" has stuck from the days when all experiments were done by directing a beam of particles onto a fixed target. Maybe a name like the "fixed target frame" would have been more appropriate for what we want to say, but we will stick to the conventional.

7.4.1 CM frame

The δ-function in Eq. (7.90) implies

$$\boldsymbol{p}_1' = -\boldsymbol{p}_2' , \tag{7.91}$$

since

$$\boldsymbol{p}_1 = -\boldsymbol{p}_2 \tag{7.92}$$

in the CM frame by definition. Then we can perform the integration over $\boldsymbol{p}_2'$, say, to write

$$\sigma = \frac{1}{64\pi^2 E_1 E_2 v_{\rm rel}} \int \frac{d^3p_1'}{E_1' E_2'} \, \delta(E_1 + E_2 - E_1' - E_2') \, \overline{|\mathcal{M}|^2} . \tag{7.93}$$

The Feynman amplitude factor has also been evaluated subject to Eqs. (7.91) and (7.92).

The remaining integration is over d^3p_1'. Noting that $\mathbf{p}_1' = \mathbf{p}_2'$, we can denote the common magnitude by $\mathbf{p}'$. The energies of the two final particles, however, can be different if their masses are different, so we keep distinguishing them by the symbols E_1' and E_2'. Then,

$$d^3p_1' = \mathbf{p}'^2 \, d\mathbf{p}' \, d\Omega = \mathbf{p}' E_1' \, dE_1' \, d\Omega , \tag{7.94}$$

where $d\Omega$ is the integration measure for the angular variables θ and φ, and we have used Eq. (7.32) to exchange the other integration variable for the energy of one of the final state particles. The general kinematical variables in the CM frame have been shown in Fig. 7.1. We now show that the magnitudes of the energies are determined by the energy in the initial particles, so we can perform the integration over it. The angle θ, however, is not determined by the initial momenta, and remains a free variable. It is called the *scattering angle*.

Squaring both sides of Eq. (7.91) and expressing the squares of the 3-momenta in terms of energy and mass, we obtain $E_1'^2 - m_1'^2 = E_2'^2 - m_2'^2$. We can thus express E_2' in terms of E_1':

$$E_2' = \sqrt{E_1'^2 - m_1'^2 + m_2'^2} . \tag{7.95}$$

It is customary to represent the total initial energy in the CM frame by the notation $\sqrt{s}$:

$$\sqrt{s} = E_1 + E_2 . \tag{7.96}$$

So the δ-function appearing in Eq. (7.93) can be written as

$$\delta\left(\sqrt{s} - E_1' - \sqrt{E_1'^2 - m_1'^2 + m_2'^2}\right) . \tag{7.97}$$

The argument of this δ-function is a function of the integration variable E_1', and so we must use Eq. (3.15) to reduce it. The argument vanishes when

$$E_1' = \frac{s + m_1'^2 - m_2'^2}{2\sqrt{s}} \equiv \omega_1 , \tag{7.98}$$

which implies, through Eq. (7.95),

$$E_2' = \frac{s - m_1'^2 + m_2'^2}{2\sqrt{s}} \equiv \omega_2 . \tag{7.99}$$

Evaluating the derivative of the argument of the δ-function of Eq. (7.97) and using Eq. (3.15), we finally obtain

$$\delta\left(\sqrt{s} - E_1' - \sqrt{E_1'^2 - m_1'^2 + m_2'^2}\right) = \delta(E_1' - \omega_1)\, \frac{\omega_2}{\sqrt{s}} . \tag{7.100}$$

We now go back to Eq. (7.93), which can be written as

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 E_1 E_2 \sqrt{s} v_{\rm rel}} \int dE_1'\ \mathbf{p}'\, \delta(E_1' - \omega_1)\, \overline{|\mathscr{M}|^2} \\ &= \frac{\mathbf{p}'}{64\pi^2 E_1 E_2 \sqrt{s} v_{\rm rel}}\, \overline{|\mathscr{M}|^2} . \end{aligned} \tag{7.101}$$

where now $\mathbf{p}'$ as well as the Feynman amplitude part have to be evaluated subject to the momentum conservation equations. Using Eq. (7.98), we find

$$\mathbf{p}' = \left(\frac{[s - (m_1' + m_2')^2]\, [s - (m_1' - m_2')^2]}{4s} \right)^{1/2} . \tag{7.102}$$

The relative velocity of the initial particles can also be expressed in terms of the variable s. Using the general definition of Eq. (7.87) and substituting the CM frame relation from Eq. (7.92), we can write

$$v_{\rm rel} = \mathbf{p}_1 \left(\frac{1}{E_1} + \frac{1}{E_2} \right) , \tag{7.103}$$

which implies that in the CM frame

$$E_1 E_2 v_{\rm rel} = \mathbf{p}\sqrt{s}\,, \tag{7.104}$$

where $\mathbf{p} = \mathbf{p}_1 = \mathbf{p}_2$. However, from the definition of s in Eq. (7.96), we can write

$$s = \left(\sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2}\right)^2 . \tag{7.105}$$

The solution to $\mathbf{p}$ is an expression similar to Eq. (7.102), with the masses of initial particles instead of the final ones.

Putting all the factors together, we can rewrite Eq. (7.101) in the final form:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left[\frac{\{s-(m_1'+m_2')^2\}\,\{s-(m_1'-m_2')^2\}}{\{s-(m_1+m_2)^2\}\,\{s-(m_1-m_2)^2\}}\right]^{1/2} \overline{|\mathscr{M}|^2}\,. \tag{7.106}$$

We emphasize again that in this expression, the Feynman amplitude factor has to be evaluated with 4-momentum conservation. For elastic scattering, this formula simplifies to

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,\overline{|\mathscr{M}|^2}\,. \tag{7.107}$$

The quantity $d\sigma/d\Omega$ is called the *differential cross-section*, since it involves derivatives of the total cross-section. In a specific problem, we need to integrate it over the angular variables to obtain the total cross-section. Notice that the azimuthal variable φ does not enter the kinematics, so the Feynman amplitude cannot depend on it. The integration over this variable will simply give a factor 2π. However, in general, the Feynman amplitude square will depend on the scattering angle θ, and therefore this dependence will have to be taken into account while performing the integration over θ.

7.4.2 Lab frame

We now turn to the other frame of interest, in which one of the initial particles is at rest. Without loss of generality, we can take this to be the first particle in the general notation for the scattering process

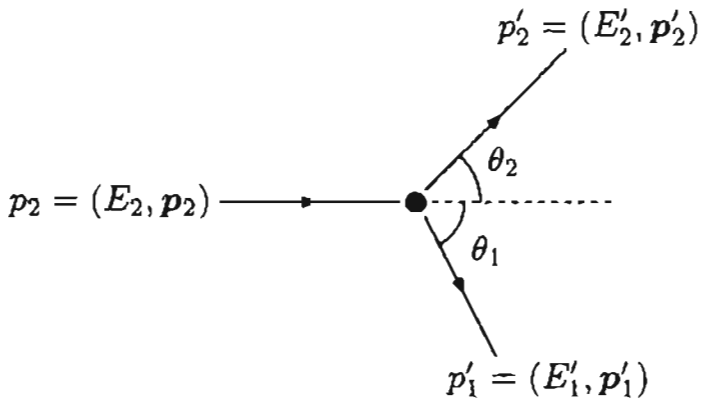


Figure 7.2: Kinematics of two particle scattering in the Lab frame. Particle 1 is at rest in this frame, shown as a blob.

given in Eq. (7.89). The general picture of the scattering process will then look like Fig. 7.2.

The most general formulas in this frame look quite cumbersome, so we are presenting the formulas for the case in which one of the particles in the initial as well as in the final state is massless. Since we have chosen the initial state as the rest frame for the particle 1, this particle cannot be massless. We will take $m_2 = 0$. This also means that $v_{\text{rel}} = 1$, since one of these initial particles is massless and the other is at rest. In the final state, without further loss of generality, we can put $m_2' = 0$.

We can still start from Eq. (7.93), except that with our assumption of $m_2' = 0$, it is more convenient to integrate over $\boldsymbol{p}_1'$ first and leave the integration over $\boldsymbol{p}_2'$ for the future. The only change in Eq. (7.93) would be to replace d^3p_1' by d^3p_2', without any other change at this stage. In this frame, the components of the momenta of different particles can be chosen as follows:

$$\begin{aligned} p_1 &= (m_1, 0, 0, 0)\,, \\ p_2 &= (E_2, 0, 0, E_2)\,, \\ p_1' &= (E_1', -\mathbf{p}_1' \sin\theta_1, 0, \mathbf{p}_1' \cos\theta_1)\,, \\ p_2' &= (E_2', E_2' \sin\theta_2, 0, E_2' \cos\theta_2)\,. \end{aligned} \tag{7.108}$$

We have chosen the z-axis in the direction of the 3-momentum of the second particle in the initial state. We certainly have the freedom to do that. And then the x-axis has been chosen so that $\boldsymbol{p}_1'$ lies in

the x-z plane. This implies that the other scattered particle must also be in the x-z plane, since no other particle has any momentum component in the y-direction and hence momentum conservation in this direction forces this component to be zero.

Applying momentum conservation conditions on the x and z components, we obtain the relations

$$\begin{aligned} \mathbf{p}_1' \sin\theta_1 - E_2' \sin\theta_2 &= 0\,, \\ \mathbf{p}_1' \cos\theta_1 + E_2' \cos\theta_2 &= E_2\,. \end{aligned} \tag{7.109}$$

These relations enable us to evaluate $\mathbf{p}_1'$ in terms of E_2'.

$$\begin{aligned} \mathbf{p}_1'^{\,2} &= (E_2 - E_2' \cos\theta_2)^2 + (E_2' \sin\theta_2)^2 \\ &= E_2^2 + E_2'^{\,2} - 2E_2 E_2' \cos\theta_2\,. \end{aligned} \tag{7.110}$$

This means

$$E_1' = \sqrt{E_2^2 + E_2'^{\,2} - 2E_2 E_2' \cos\theta_2 + m_1'^{\,2}}\,. \tag{7.111}$$

Writing

$$d^3p_2' = E_2'^{\,2}\, dE_2'\, d\Omega_2 \tag{7.112}$$

in analogy with Eq. (7.94), we can now express the differential cross section in the following form:

$$\frac{d\sigma}{d\Omega_2} = \frac{1}{64\pi^2 m_1 E_2} \int dE_2' \, \frac{E_2'}{E_1'} \, \delta(m_1 + E_2 - E_1' - E_2') \, \overline{|\mathscr{M}|^2}. \tag{7.113}$$

where E_1' is related to E_2' by Eq. (7.111).

The argument of the δ-function is a function of the integration variable E_2', so we must again use Eq. (3.15) to simplify it. The argument vanishes when

$$(m_1 + E_2 - E_2')^2 = E_2^2 + E_2'^{\,2} - 2E_2 E_2' \cos\theta_2 + m_1'^{\,2}\,, \tag{7.114}$$

i.e., when

$$E_2' = \frac{m_1^2 + 2m_1 E_2 - m_1'^{\,2}}{2(m_1 + E_2 - E_2 \cos\theta_2)} \equiv \omega_2\,. \tag{7.115}$$

Thus,

$$\delta(m_1 + E_2 - E_1' - E_2') = \delta(E_2' - \omega_2) \left| 1 + \frac{dE_1'}{dE_2'} \right|^{-1}_{E_2' = \omega_2}\,. \tag{7.116}$$

Using the expression for E_1' from Eq. (7.111), we obtain

$$\left|1+\frac{dE_1'}{dE_2'}\right|_{E_2'=\omega_2} = \left|1+\frac{E_2'-E_2\cos\theta_2}{E_1'}\right|_{E_2'=\omega_2} = \frac{m_1+E_2-E_2\cos\theta_2}{\omega_1}, \qquad (7.117)$$

where we have used the δ-function to replace $E_1'+E_2'$ by m_1+E_2, and written ω_1 for the value of E_1' when $E_2'=\omega_2$. Substituting the δ-function in Eq. (7.113) and performing the integration over E_2', we obtain

$$\frac{d\sigma}{d\Omega_2} = \frac{1}{128\pi^2 m_1 E_2}\,\frac{m_1^2+2m_1E_2-m_1'^2}{(m_1+E_2-E_2\cos\theta_2)^2}\,\overline{|\mathscr{M}|^2}. \qquad (7.118)$$

Once again, of course, it has to be remembered that the quantity $\overline{|\mathscr{M}|^2}$ has to be evaluated subject to the restrictions imposed by momentum conservation.

□ **Exercise 7.7** Consider the decay of a particle of mass M into two particles of masses m_1 and m_2. Following an analysis similar to what was done in this section, show that the differential decay rate in the rest frame of the decaying particle is given by

$$\frac{d\Gamma}{d\Omega} = \frac{\overline{|\mathscr{M}|^2}}{64\pi^2 M}\sqrt{\left[1-\left(\frac{m_1+m_2}{M}\right)^2\right]\left[1-\left(\frac{m_1-m_2}{M}\right)^2\right]}. \qquad (7.119)$$

Use this directly to obtain the decay rate derived in Eq. (7.34).

7.5 Inelastic scattering with 4-fermion interaction

In this section, we present an example of calculation of scattering cross sections, viz, that of neutrino-electron scattering. Many more examples of scattering processes will be discussed in the later chapters, particularly in Ch. 9 and Ch. 15.

Many kinds of neutrino-electron scattering processes can be discussed with 4-fermion interactions. For example, one can discuss $\nu_e e \to \nu_e e$, or $\nu_\mu e \to \nu_\mu e$. In such cases, the initial and the final states have the same particles, and these processes are instances of elastic scattering. Here, we discuss another process, where the scattering is inelastic.

The process we have in mind is:

$$e^-(p) + \nu_\mu(k) \to \mu^-(p') + \nu_e(k')\,, \tag{7.120}$$

where, as usual, p, k etc. are the momenta of various particles. The interaction Lagrangian describing the process is

$$\mathscr{L}_{\rm int} = \frac{G_F}{\sqrt{2}}\,\overline{\psi}_{(\nu_e)}\gamma^\lambda(1-\gamma_5)\psi_{(e)}\;\overline{\psi}_{(\mu)}\gamma_\lambda(1-\gamma_5)\psi_{(\nu_\mu)}\,, \tag{7.121}$$

which is really the hermitian conjugate of the interaction term given in Eq. (7.36) to describe muon decay. The interaction Hamiltonian is the negative of the interaction Lagrangian, and so the Feynman amplitude can be written as

$$\mathscr{M}_{fi} = \frac{G_F}{\sqrt{2}}\,\overline{u}_{(\nu_e)}(k')\gamma^\lambda(1-\gamma_5)u_{(e)}(p)\;\overline{u}_{(\mu)}(p')\gamma_\lambda(1-\gamma_5)u_{(\nu_\mu)}(k)\,. \tag{7.122}$$

We will square this amplitude, with a sum over final spins and an average over initial spins. For the averaging, one peculiarity of neutrinos has to be noted. Although they are spin-$\frac{1}{2}$ particles, they appear to come only in one helicity state, with helicity -1. Thus, averaging over initial spins will involve a division by 2 because of the electron spin only. There will be no corresponding factor from neutrino spin states. So, in the general formulas for cross sections derived in §7.4, we should put

$$\begin{aligned}\overline{|\mathscr{M}|^2} &= \frac{1}{2}\sum_{\rm spin}|\mathscr{M}|^2 \\ &= \frac{G_F^2}{4}\,{\rm Tr}\left[(\not{p}+m_e)\gamma^\rho(1-\gamma_5)\not{k}'\gamma^\lambda(1-\gamma_5)\right] \\ &\qquad \times {\rm Tr}\left[\not{k}\gamma_\rho(1-\gamma_5)(\not{p}'+m_\mu)\gamma_\lambda(1-\gamma_5)\right]\,,\end{aligned} \tag{7.123}$$

using the techniques described in §7.2.2. Apart from some minor changes in notation, this is the same as the expression as in Eq. (7.41). Since we discussed the traces in detail there, we will skip the calculations here and simply write down the final result, which is:

$$\overline{|\mathscr{M}|^2} = 64G_F^2\;p\cdot k\;p'\cdot k'\,. \tag{7.124}$$

We can now use this to evaluate the cross-section in either the CM frame or the Lab frame. Let us discuss them one by one. We can put $m_1 = m_e$ and $m_1' = m_\mu$, whereas $m_2 = m_2' = 0$ since the neutrinos are massless.

7.5.1 Cross-section in CM frame

In the CM frame where $\boldsymbol{p} = -\boldsymbol{k}$, we can use Eq. (7.106) for the differential cross-section to obtain

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left(\frac{s-m_\mu^2}{s-m_e^2}\right)\overline{|\mathscr{M}|^2}\,. \tag{7.125}$$

To complete the calculation, we need to express the square of the Feynman amplitude in terms of the kinematical variables in the CM frame. We first note that $\mathbf{p} = \mathbf{k}$. The energy of the initial neutrino is then $\mathbf{p}$ as well, since it is massless. The energy of the incoming electron will be denoted by E. Adapting the general formula of Eq. (7.105) for the present purpose, we can write

$$\mathbf{p} = \frac{s-m_e^2}{2\sqrt{s}}\,. \tag{7.126}$$

Therefore,

$$E = \sqrt{\mathbf{p}^2 + m_e^2} = \frac{s+m_e^2}{2\sqrt{s}}\,. \tag{7.127}$$

Then

$$p\cdot k = E\mathbf{p} + \mathbf{p}^2 = \frac{1}{2}(s-m_e^2)\,. \tag{7.128}$$

The quantity $p'\cdot k'$, which also appears in the expression of Eq. (7.124), can be evaluated by starting from the momentum conservation equation $p + k = p' + k'$, and squaring both sides. This gives, since $k^2 = k'^2 = 0$,

$$p'\cdot k' = p\cdot k - \frac{1}{2}(m_\mu^2 - m_e^2) = \frac{1}{2}(s-m_\mu^2)\,. \tag{7.129}$$

Putting these back into Eq. (7.124) and using Eq. (7.125), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{4\pi^2}\,\frac{(s-m_\mu^2)^2}{s}\,, \tag{7.130}$$

which is independent of the scattering angle in this case. Thus, integration over the angular variables is trivial, and it yields a factor 4π. The total cross-section is given by

$$\sigma = \frac{G_F^2}{\pi}\,\frac{(s-m_\mu^2)^2}{s}\,. \tag{7.131}$$

For $s \gg m_\mu^2$, the total cross-section is proportional to s.

7.5.2 Cross-section in Lab frame

To evaluate the same cross-section in the Lab frame, we can use the general formulas of §7.4.2 since we have one massless particle in both the initial and the final states. In this frame,

$$p \cdot k = m_e \omega \,, \tag{7.132}$$

$\omega = \mathbf{k}$ being the energy of the incoming neutrino. Then, using the first part of Eq. (7.129), we obtain

$$p' \cdot k' = \frac{1}{2}\left(m_e^2 + 2m_e\omega - m_\mu^2\right) . \tag{7.133}$$

Putting these in the expression for $\overline{|\mathscr{M}|^2}$ appearing in Eq. (7.124) and using the expression for the differential cross-section in Eq. (7.118), we obtain

$$\frac{d\sigma}{d\Omega_2} = \frac{G_F^2}{4\pi^2}\left(\frac{m_e^2 + 2m_e\omega - m_\mu^2}{m_e + \omega - \omega\cos\theta_2}\right)^2 . \tag{7.134}$$

Unlike in the CM frame, the differential cross-section depends on the scattering angle in this frame.

To obtain the total cross-section, we can integrate over the angular variables. The integration over the azimuthal angle gives a factor of 2π, so we obtain

$$\begin{aligned}\sigma &= \frac{G_F^2}{2\pi}\int_{-1}^{1} d(\cos\theta_2)\left(\frac{m_e^2 + 2m_e\omega - m_\mu^2}{m_e + \omega - \omega\cos\theta_2}\right)^2 \\ &= \frac{G_F^2}{\pi}\frac{(m_e^2 + 2m_e\omega - m_\mu^2)^2}{m_e^2 + 2m_e\omega} .\end{aligned} \tag{7.135}$$

This is the same as the total cross-section obtained in the CM frame for

$$s = m_e^2 + 2m_e\omega \,. \tag{7.136}$$

This is not a coincidence. It will be discussed in more detail in the next section.

□ **Exercise 7.8** Show that the CM frame for the ν_μ-e scattering moves with a speed $v = \omega/(m_e + \omega)$ with respect to the rest frame of the electron. Calculate the energy of both particles in this frame. If they are denoted by E_v and ω_v, show that $s = (E_v + \omega_v)^2 = m_e^2 + 2m_e\omega$.

□ **Exercise 7.9** In the Lab frame in which the initial electron is at rest, calculate the minimum (or threshold) energy that the incoming neutrino must possess in order to initiate the process described here. Use $m_e = 0.511\,\text{MeV}$ and $m_\mu = 106\,\text{MeV}$. [Hint: The energy E_2', given in Eq. (7.115), must be positive.]

7.6 Mandelstam variables

Let us go back to the general definition of the scattering cross-section in Eq. (7.88). The absolute square of the Feynman amplitude is a Lorentz invariant quantity. So is the energy-momentum conserving δ-function. The integration measures are also Lorentz invariant, as argued in Eq. (7.53). With v_{rel} as defined in Eq. (7.86), the product $v_{\text{rel}} E_1 E_2$ is also Lorentz invariant. It then follows that the total cross section is Lorentz invariant. Of course, differential cross-sections need not be.

Since the total cross-section is invariant, it would be elegant to present the formulas in a notation which is explicitly Lorentz invariant. Instead of individual energies and momenta, we can write the cross-section in terms of their Lorentz invariant combinations. For a 2-to-2 scattering, the invariant combinations that one uses are called Mandelstam variables. Let us first define them, in the general notation of scattering variables introduced in Eq. (7.89):

$$\begin{aligned} s &= (p_1 + p_2)^2\,, \\ t &= (p_1 - p_1')^2\,, \\ u &= (p_1 - p_2')^2 \end{aligned} \tag{7.137}$$

Of these three, the quantity s was introduced before in specific frames. The general definition given here reduces to the definition of Eq. (7.96) for the CM frame since $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$ in that frame. Also, in the Lab frame in which one of the initial particles is at rest, it reduces to the expression of Eq. (7.136).

The definition of the other two Mandelstam variables, viz., t and u, has some ambiguity, since there is no universal way of assigning which one of the final particles is the first one, and which one is the second. Thus, two persons may disagree about what is t and what is u in a given process, but it is not important. If we state our assignments clearly, there would be no ambiguity.

Let us, however, try to convince ourselves that the three variables are sufficient to describe the cross-section for 2-to-2 scattering. There are four particles, but because of 4-momentum conservation, the 4-momenta of three of them are independent. From three momenta, we can make six invariants, viz., we can contract each momenta with itself, and with one another. Thus we will have six different scalar products. However, four combinations of these will give the masses of the four particles involved. Thus, we are left with only two scalar parameters. And we have defined three Mandelstam variables. This shows that the Mandelstam variables are not only sufficient, in fact there is some redundancy in their definition. In fact, it is straight forward to verify that

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2 \,. \tag{7.138}$$

□ **Exercise 7.10** If the decay rate of a particle of mass M is Γ in its rest frame and Γ' in a frame where its energy is E, argue from Eq. (7.8) that

$$\Gamma' = \frac{M}{E}\,\Gamma \tag{7.139}$$

irrespective of the interactions.

Chapter 8

Quantization of the electromagnetic field

We have already discussed how to quantize fields with spin-0 and spin-$\frac{1}{2}$. In this chapter, we describe how to quantize spin-1 fields. A familiar example of spin-1 particles is the photon, which is the quantum of the electromagnetic field. Let us now see how the photon appears through quantization of this field. This will help us see the connection between the classical theory and its quantized version. To set up the notation, we will begin with a brief summary of classical electromagnetic theory.

8.1 Classical theory of electromagnetic fields

The classical theory of electromagnetic fields is based on the Maxwell equations. In a notation which is not manifestly covariant, these equations can be written as

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \rho\,, \\ \nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{j}\,, \end{aligned} \tag{8.1}$$

in Heaviside-Lorentz units with $c = 1$. Here ρ is called the *charge density* and $\boldsymbol{j}$ the *current density*, and the components of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ are constrained by two further equations:

$$\begin{aligned} \nabla \cdot \boldsymbol{B} &= 0\,, \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}\,. \end{aligned} \tag{8.2}$$

These last two equations involve six quantities, viz., the components of the 3-vectors $\boldsymbol{E}$ and $\boldsymbol{B}$. They can be expressed in terms of four quantities, viz., the components of a 3-vector $\boldsymbol{A}$ and a scalar quantity φ:

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A}\,, \\ \boldsymbol{E} &= -\boldsymbol{\nabla}\varphi - \frac{\partial \boldsymbol{A}}{\partial t}\,. \end{aligned} \tag{8.3}$$

In fact, these four quantities transform like the components of a 4-vector A, i.e.,

$$A^\mu \equiv (A^0, \boldsymbol{A}) = (\varphi, \boldsymbol{A})\,. \tag{8.4}$$

In terms of these quantities, one can now write Eq. (8.3) in a manifestly covariant form:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{8.5}$$

where the components of the field-strength tensor $F_{\mu\nu}$ are the components of the electric and magnetic fields:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}\,. \tag{8.6}$$

$F_{\mu\nu}$ is called the *field strength tensor*, while A^μ is called the *potential*. The components of $F^{\mu\nu}$ can be obtained by replacing E^i by $-E^i$ in Eq. (8.6). The two homogeneous Maxwell equations given in Eq. (8.2) can now be written in the form

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0\,. \tag{8.7}$$

On the other hand, the inhomogeneous Maxwell equations, given in Eq. (8.1), can be written as

$$\partial_\mu F^{\mu\nu} = j^\nu\,, \tag{8.8}$$

where j^ν is a 4-vector which incorporates the sources, i.e., the charge density and the current density:

$$j^\mu \equiv (j^0, \boldsymbol{j}) = (\rho, \boldsymbol{j})\,. \tag{8.9}$$

The Lagrangian formulation of this theory can be performed if we start from

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{8.10}$$

The Euler-Lagrange equations obtained from this Lagrangian by varying A^μ are precisely those given in Eq. (8.8). The current density j^μ, which comes from external sources, depends on the field operators of these source fields. For example, if the sources are electrons, we should have $j^\mu = -e\overline{\psi}\gamma^\mu\psi$. In field theoretic language, we can say that this is an interaction term. We will discuss these terms in Ch. 9. In this chapter, we discuss only the pure electromagnetic field, which can be described by

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.11}$$

For future purposes, let us also note that $F_{\mu\nu}$ is invariant under the redefinitions of A_μ which are of the form

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\theta\,, \tag{8.12}$$

where θ is any differentiable function of space-time. Such transformations can affect the values of A^μ at each space-time point differently, and are called *local transformations*. In contrast, for the phase rotations of fermion fields or of complex scalar fields, which we discussed in Eqs. (3.54) and (4.91), the transformation was the same for all space-time points. This type of transformations are called *global transformations*.

Under the transformation (8.12), the Lagrangian of Eq. (8.10) changes by

$$\delta\mathscr{L} = -j^\mu\partial_\mu\theta = -\partial_\mu(j^\mu\theta) + (\partial_\mu j^\mu)\theta \tag{8.13}$$

If the current j^μ is a conserved current, $\partial_\mu j^\mu = 0$, we find that the Lagrangian changes by a total divergence. It follows that the dynamics of the electromagnetic field is invariant under (8.12) even when it is coupled to an external current, as long as the current is conserved. This derivation assumes that j^μ is independent of A^μ This assumption may not be true in general, as we found for scalar electrodynamics in Ex. 2.9 (p 26). Even then the Lagrangian may be invariant, as it was there, but the proof is a little more complicated.

8.2 Problems with quantization

There are many ways to see that the classical theory of electromagnetic fields, outlined above, cannot be directly quantized. One way is to convince oneself that the propagator of such a field does not exist. For this, let us go back to the equation of motion, Eq. (8.8). This can be rewritten, using the definition of the field tensor from Eq. (8.5), in the following form:

$$\partial_\mu \left(\partial^\mu A^\lambda - \partial^\lambda A^\mu \right) = j^\lambda \,, \tag{8.14}$$

or

$$\left(g^{\lambda\mu} \Box - \partial^\lambda \partial^\mu \right) A_\mu = j^\lambda \tag{8.15}$$

The equation for the corresponding Green's function will be

$$\left(g^{\lambda\mu} \Box - \partial^\lambda \partial^\mu \right) D_{\mu\nu}(x - x') = g^\lambda{}_\nu \, \delta^4(x - x') \,, \tag{8.16}$$

where all the derivatives are with respect to the co-ordinate x. The Fourier transform of this equation will read

$$- \left(g^{\lambda\mu} k^2 - k^\lambda k^\mu \right) D_{\mu\nu}(k) = g^\lambda{}_\nu \tag{8.17}$$

The Fourier transform of the proposed propagator, $D_{\mu\nu}(k)$, must be a rank-2 tensor which depends only on the 4-vector k. Thus, its most general form can be written as

$$D_{\mu\nu}(k) = a g_{\mu\nu} + b k_\mu k_\nu \,, \tag{8.18}$$

where a and b are Lorentz-invariant quantities. Putting this form back into Eq. (8.17), we see that the terms involving b cancel out. This is the first indication that the propagator does not exist, because b is indeterminable. Moreover, we are left with the following equation involving a:

$$-a k^2 g^\lambda{}_\nu + a k^\lambda k_\nu = g^\lambda{}_\nu \tag{8.19}$$

If we now want to equate the coefficients of $g^\lambda{}_\nu$ on both sides, we would get $a = -1/k^2$, but the $k_\lambda k^\nu$ term gives $a = 0$. In short, this equation cannot be solved. The propagator does not exist.

The basic reason for the non-existence of the propagator is the following. The momenta conjugate to the A^μ are given by

$$\Pi^\mu = \frac{\delta L}{\delta \dot{A}_\mu} = \frac{\delta L}{\delta(\partial_0 A_\mu)} = F^{\mu 0} . \tag{8.20}$$

It follows that $\Pi^0 = 0$. One of the canonical momenta does not exist. The equations defining the canonical momenta cannot therefore be inverted to express the quantities $\dot{A}_\mu$ in terms of the momenta. Accordingly, the Hamiltonian formulation does not exist.

The deeper reason for this fact can be understood by looking at the Maxwell equations in Eq. (8.1). The field variables are the six components of the 3-vectors $\boldsymbol{B}$ and $\boldsymbol{E}$, as mentioned earlier. However, Eq. (8.2) introduces four constraints on them, so that there are really two independent degrees of freedom in the electromagnetic field. We are trying to represent these by the 4-vector A^μ. So, all the components of A^μ are not independent. We have never encountered such a situation while talking about scalar fields or Dirac fields. Indeed, a Dirac field also had four components, but all of them were independent. For A^μ, only two of the four components are independent, and this is the root of the problem. We will have to somehow deal with this problem of redundancy before we can quantize the theory. Finally, although the field A^μ is called the potential in classical physics, we shall refer to it as the photon field.

□ **Exercise 8.1** Suppose, instead of the photon field, we are considering the theory of a massive vector field, given by the Proca Lagrangian

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A^\mu A_\mu - A_\mu j^\mu . \tag{8.21}$$

Find the equation of motion. Show that in this case, there is no problem with defining the propagator, and one obtains

$$D_{\mu\nu}(k) = \frac{1}{k^2 - M^2} \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right) . \tag{8.22}$$

8.3 Modifying the classical Lagrangian

The photon is known to be massless to a very high degree of accuracy. So we will not try to add a mass term as in Eq. (8.21) just to

define the propagator. Such a mass term is also not invariant under the local transformation noted in Eq. (8.12). Invariance under this transformation means that given any electric and magnetic field, there is some freedom in defining what the A^μ should be. Therefore, if we try to find $A^\mu(x)$ at any space-time point x by operating on the sources $j^\mu(x')$ by a Green's function, the Green's function needs to be able to find all the $A^\mu(x)$ which are related by Eq. (8.12). Such a Green's function cannot exist, as we have seen in the previous section. This is a classical problem, not a quantum one. A first step towards a resolution is to look for a representative $A^\mu(x)$ from each (infinite) set of $A^\mu(x)$ connected by Eq. (8.12). This representative $A^\mu(x)$ can be chosen by imposing a condition on $A^\mu(x)$, and the procedure is called *gauge-fixing*. Examples of gauge-fixing relations are the Lorenz gauge, $\partial_\mu A^\mu = 0$, the Coulomb gauge, $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, the axial gauge, $A_3 = 0$, the temporal gauge, $A_0 = 0$. Different choices of the gauge-fixing relation (also called gauge choices) are useful in different situations. We are interested in a Lagrangian formulation, so we shall incorporate the gauge-fixing procedure directly into the Lagrangian. We shall use a generalization of the Lorenz gauge, which is the most commonly used manifestly Lorentz-covariant gauge-fixing relation.

Let us start this procedure by rewriting the action arising from the Lagrangian of Eq. (8.11).

$$\begin{aligned} \mathscr{A} &= -\frac{1}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{2}\int d^4x\ [(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)]\ . \end{aligned} \tag{8.23}$$

The second term can be rewritten as

$$\begin{aligned} \int d^4x\ [(\partial_\mu A_\nu)(\partial^\nu A^\mu)] &= \int d^4x\ [\partial_\mu(A_\nu\partial^\nu A^\mu) - A_\nu(\partial_\mu\partial^\nu A^\mu)] \\ &= \int d^4x\ \Big[\partial_\mu(A_\nu\partial^\nu A^\mu) - \partial^\nu(A_\nu\partial_\mu A^\mu) \\ &\qquad + (\partial^\nu A_\nu)(\partial_\mu A^\mu)\Big]. \end{aligned} \tag{8.24}$$

The total divergence terms, on integration, will produce surface terms which are assumed to vanish on the boundary of the 4-dimensional space-time, which is at infinity. Thus we are left with

$$\mathscr{A} = -\frac{1}{2}\int d^4x\ \Big[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A^\mu)^2\Big]\ . \tag{8.25}$$

Of course, this is still the same action. We have not made any changes whatsoever. So all the problems that we mentioned earlier still remain with us.

As we mentioned, the problem originated because we had too much freedom in defining the A^μ. We need to sacrifice some of this freedom. We do this by introducing another term in the Lagrangian, which is

$$\mathscr{L}_{\rm GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \,. \tag{8.26}$$

This is called the gauge-fixing term, which is why we have put a subscript "GF" on this term. The total action, after the inclusion of this term, is

$$\mathscr{A} = -\frac{1}{2}\int d^4x \left[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - \left(1-\frac{1}{\xi}\right)(\partial_\mu A^\mu)^2\right]. \tag{8.27}$$

This is for the free photon field. In presence of sources, we can simply augment it by the $j_\mu A^\mu$ term as in Eq. (8.10). The equations of motion that would then follow from this action is:

$$\Box A^\nu - \left(1-\frac{1}{\xi}\right)\partial^\nu(\partial_\mu A^\mu) = j^\nu \,, \tag{8.28}$$

instead of Eq. (8.8) which can be obtained from this equation by letting ξ go to infinity.

□ **Exercise 8.2** If $\partial_\mu A^\mu(x) = f(x)$, construct $\theta(x)$ such that $A'_\mu = A_\mu + \partial_\mu\theta$ satisfies $\partial_\mu A'^\mu = 0$. [Hint: Use scalar propagator.]

It is a fair question to ask who gave us the right to change Maxwell's equations, which has been verified so well in experiments. We can argue, however, that we have not really changed the Maxwell equations as they would appear in terms of the field tensor $F_{\mu\nu}$. What we are changing, or rather restricting, is the definition of $F_{\mu\nu}$ in terms of A_μ. Previously, given any $F_{\mu\nu}$, we could have taken any A_μ satisfying Eq. (8.5). We can no longer do that. In fact, if we use the freedom of the local transformation of Eq. (8.12) to define the A^μ in such a way that

$$\partial_\mu A^\mu = 0 \,, \tag{8.29}$$

the gauge-fixing term vanishes, and we have done no offense to the Maxwell equations. And this can be done, at least classically, which

can be seen from Eq. (8.28). Since the current j^μ is conserved, i.e., $\partial_\mu j^\mu = 0$, taking the 4-divergence of Eq. (8.28) we obtain

$$\Box(\partial_\mu A^\mu) = 0\,. \tag{8.30}$$

We can now argue that appropriate boundary conditions can be imposed on A^μ so that Eq. (8.29) is satisfied everywhere. In other words, we can always transform A_μ according to Eq. (8.12) so that the new A_μ satisfy Eq. (8.29). Thus, in order not to disturb Maxwell equations, we are now restricted to use only those A^μ's which satisfy Eq. (8.29).

In passing, we note that the canonical momenta are now given by

$$\begin{aligned}\Pi^\mu &= \frac{\delta L}{\delta(\partial_0 A_\mu)} \\ &= -\dot{A}^\mu + \left(1 - \frac{1}{\xi}\right) g^{\mu 0}\partial_\nu A^\nu\,.\end{aligned} \tag{8.31}$$

Obviously, Π^0 no longer vanishes identically. But we still have a problem if we try to canonically quantize the theory. If we try to mimic the technique employed for scalar fields and write

$$[A_\mu(t,\boldsymbol{x}), \Pi^\nu(t,\boldsymbol{y})]_- = i\delta^\nu_\mu \delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{8.32}$$

it will imply

$$\left[\frac{\partial}{\partial x^\mu}A^\mu(t,\boldsymbol{x}), \Pi_\nu(t,\boldsymbol{y})\right]_- = i\frac{\partial}{\partial x^\nu}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \neq 0\,. \tag{8.33}$$

Thus, if $\partial_\mu A^\mu$ vanishes everywhere as we argued a little while ago, this commutation relation cannot be imposed. However, we shall worry about this later. Our first priority now is to get a sensible propagator.

8.4 Propagator

Before introducing the gauge fixing term, we showed that the propagator could not be defined. Let us revisit this issue now, aided with the gauge fixing term.

The equation of motion, including the gauge fixing term, was given in Eq. (8.28). We can rewrite it as

$$\left[g^{\lambda\mu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\lambda\partial^\mu\right] A_\mu = j^\lambda \,. \tag{8.34}$$

The equation for the Green's function will be

$$\left[g^{\lambda\mu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\lambda\partial^\mu\right] D_{\mu\nu}(x - x') = g^\lambda{}_\nu \delta^4(x - x') \,. \tag{8.35}$$

Taking the Fourier transform of this equation, we obtain

$$-\left[g^{\lambda\mu}k^2 - \left(1 - \frac{1}{\xi}\right)k^\lambda k^\mu\right] D_{\mu\nu}(k) = g^\lambda{}_\nu \,. \tag{8.36}$$

The general tensorial form of the propagator is still given by Eq. (8.18). Putting this form back into Eq. (8.36), we obtain

$$\begin{aligned} g^\lambda{}_\nu &= -\left[g^{\lambda\mu}k^2 - \left(1 - \frac{1}{\xi}\right)k^\lambda k^\mu\right](a g_{\mu\nu} + b k_\mu k_\nu) \\ &= -ak^2 g^\lambda{}_\nu + \left[a\left(1 - \frac{1}{\xi}\right) - \frac{1}{\xi}bk^2\right] k^\lambda k_\nu \,. \end{aligned} \tag{8.37}$$

This can now be solved, and the solution is

$$a = -\frac{1}{k^2}, \qquad b = \frac{1 - \xi}{k^4} \,. \tag{8.38}$$

The propagator is then given by

$$D_{\mu\nu}(k) = -\frac{1}{k^2}\left[g_{\mu\nu} - (1 - \xi)\frac{k_\mu k_\nu}{k^2}\right] \,. \tag{8.39}$$

Like the propagator for the scalar and fermion fields, there is a problem if we try to find the propagator in co-ordinate space. The problem can be solved in the same way, so that we get the Feynman propagator:

$$D_{\mu\nu}(k) = -\frac{1}{k^2 + i\varepsilon}\left[g_{\mu\nu} - (1 - \xi)\frac{k_\mu k_\nu}{k^2}\right] \,. \tag{8.40}$$

Although the problem of non-existence of a gauge-invariant propagator is a classical one, this particular propagator lies outside the purview of classical physics. One way to see this is to note that this propagator is complex, due to the $i\varepsilon$ term in the denominator. That

would not be allowed in classical physics, where a real source leads to a real field. On the other hand in quantum physics only observables need be Hermitian (with real eigenvalues), and we have a certain leeway as a result.

It is natural to ask at this point if the propagator of Eq. (8.40) really has any meaning at all. It depends on an arbitrary parameter ξ which was put in to modify the action. This parameter ξ labels a class of theories which are all classically equivalent to Maxwell's electrodynamics *in the gauge* $\partial_\mu A^\mu = 0$. Have we then found a class of propagators for these theories which will give different results that agree only in that specific gauge? More specifically, if we calculate a Feynman diagram involving an internal photon line, will the use of this propagator lead to a result that depends on ξ and can therefore have (almost) any value?

But there is no need to worry. In fact, in the calculation of S-matrix elements, the ξ-dependence always cancels out. The result will be independent of ξ. We will show this explicitly with some examples in Ch. 9. In practical calculations, this fact can be utilized in two ways. Either we can evaluate diagrams with this general propagator and check whether the parameter ξ really cancels, which will give a good cross check for our calculations. Or we can make the calculations simpler by choosing any value of ξ that is convenient. Some common choices are:

$$\xi = 1 \Rightarrow iD_{\mu\nu}(k) = -\frac{ig_{\mu\nu}}{k^2 + i\varepsilon}\,, \tag{8.41}$$

$$\xi = 0 \Rightarrow iD_{\mu\nu}(k) = -\frac{i}{k^2 + i\varepsilon}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right]. \tag{8.42}$$

The first one is usually referred to as the Feynman-'t Hooft gauge, whose advantage is that the propagator is very simple, contains just one term. The second one is usually called the Landau gauge, which satisfies the relation $k^\mu D_{\mu\nu}(k) = 0$, which simplifies the calculations in some cases.

We shall usually work in the $\xi = 1$ gauge and explicitly indicate if we change to another gauge choice. In this gauge, the action reads

$$\mathscr{A} = -\frac{1}{2}\int d^4x\, (\partial_\mu A_\nu)(\partial^\mu A^\nu)\,, \tag{8.43}$$

and the equations of motion are

$$\Box A^\mu = j^\mu\,, \tag{8.44}$$

which look exactly like the equations for four massless scalar fields. We can therefore utilize our knowledge of scalar fields and decompose A^μ.

8.5 Fourier decomposition of the field

Mimicking the Fourier expansions in the cases of scalar and fermionic fields, we can write for the photon field

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} \left[\epsilon_r^\mu(k) a_r(k) e^{-ik\cdot x} + \epsilon_r^{*\mu}(k) a_r^\dagger(k) e^{ik\cdot x}\right], \tag{8.45}$$

where

$$\omega_k = \mathbf{k}, \tag{8.46}$$

and ϵ_r^μ constitute a set of four linearly independent 4-vectors which are called the *polarization vectors*. The Lorentz transformation properties of A^μ are carried by these polarization vectors since the other factors, as in the scalar case, transform like an overall Lorentz scalar.

The polarization vectors ϵ_r^μ satisfy the following orthonormality relations:

$$\epsilon_r^\mu \epsilon_{s\mu}^* = -\zeta_r \delta_{rs}, \tag{8.47}$$

where sum on the repeated index r will *not* be assumed unless explicitly indicated, and the quantities ζ_r are defined as

$$\zeta_0 = -1, \qquad \zeta_1 = \zeta_2 = \zeta_3 = +1. \tag{8.48}$$

These vectors also satisfy a completeness relation, given by

$$\sum_{r=0}^{3} \zeta_r \epsilon_r^\mu \epsilon_r^{*\nu} = -g^{\mu\nu}. \tag{8.49}$$

In order to understand the physical content of these relations, let us consider a definite choice of the polarization vectors. Let n^μ be an arbitrary time-like vector satisfying $n^\mu n_\mu = 1$, $n^0 > 0$. We shall set ϵ_0^μ, called the *scalar* polarization vector, to equal n^μ. Next we choose

ϵ_3^μ, called the *longitudinal* polarization vector, in the n-k plane such that $\epsilon_{3\mu} n^\mu = 0$, $\epsilon_3 \cdot \epsilon_3 = -1$. It follows that

$$\epsilon_3^\mu(k) = \frac{k^\mu - (k \cdot n) n^\mu}{\sqrt{(k \cdot n)^2 - k^2}} \,. \tag{8.50}$$

□ **Exercise 8.3** Show, without assuming $k^2 = 0$, that Eq. (8.50) gives the unique vector in the n-k plane satisfying the conditions $\epsilon_{3\mu} n^\mu = 0$ and $\epsilon_3 \cdot \epsilon_3 = -1$.

Now we choose the remaining pair of polarization vectors $\epsilon_1^\mu(k)$ and $\epsilon_2^\mu(k)$ to be in the plane orthogonal to the n-k plane (which is also the ϵ_0-ϵ_3 plane by our construction), such that

$$\epsilon_r^\mu(k) \epsilon_{s\mu}^*(k) = -\delta_{rs}, \qquad r, s = 1, 2 \,. \tag{8.51}$$

These last two are called the *transverse* polarization vectors. In the frame where the third axis is chosen along the direction of propagation (i.e. along $\boldsymbol{k}$) and $n^\mu = (1, 0, 0, 0)$, we find from Eq. (8.50) that the longitudinal polarization vector ϵ_3^μ is also along the third axis, and we can write

$$\begin{aligned} \epsilon_0^\mu &= (1, 0, 0, 0) \,, \\ \epsilon_1^\mu &= (0, 1, 0, 0) \,, \\ \epsilon_2^\mu &= (0, 0, 1, 0) \,, \\ \epsilon_3^\mu &= (0, 0, 0, 1) \,. \end{aligned} \tag{8.52}$$

The polarization vector ϵ_0^μ is timelike, and is usually called the scalar polarization since it is a scalar so far as the 3-dimensional space is concerned. The other three are spacelike. Of them, there is one linear combination which would be in the direction of the 3-momentum $\boldsymbol{k}$. This is the longitudinal polarization mode. The other two are the transverse modes. Since $\boldsymbol{k}$ is chosen along the z-direction, ϵ_3^μ is the longitudinal mode, whereas ϵ_1^μ and ϵ_2^μ are transverse modes.

Of course, these choices are not unique. In general, one can take any four vectors satisfying Eqs. (8.47) and (8.49). For example, in some situations it is useful to replace ϵ_1^μ and ϵ_2^μ by

$$\begin{aligned} \epsilon_L^\mu &= (0, 1, i, 0)/\sqrt{2} \,, \\ \epsilon_R^\mu &= (0, 1, -i, 0)/\sqrt{2} \,, \end{aligned} \tag{8.53}$$

which represent left- and right circularly polarized radiation for propagation along the z-direction. The choice of ϵ_1^μ and ϵ_2^μ, on the other hand, represents plane polarized radiation. One can also choose something more general than those given in Eq. (8.53), e.g.:

$$\begin{aligned} \epsilon_L^\mu &= (0, \cos\theta, i\sin\theta, 0)\,, \\ \epsilon_R^\mu &= (0, \sin\theta, -i\cos\theta, 0)\,, \end{aligned} \tag{8.54}$$

for an arbitrary parameter θ, which represent elliptical polarization states.

Quantization is based on the commutation relation given in Eq. (8.32), along with the relations

$$\begin{aligned} [A_\mu(t,\boldsymbol{x}), A_\nu(t,\boldsymbol{y})]_- &= 0\,, \\ [\Pi_\mu(t,\boldsymbol{x}), \Pi_\nu(t,\boldsymbol{y})]_- &= 0\,. \end{aligned} \tag{8.55}$$

On the operators $a_r(k)$'s and their hermitian conjugates, this implies the relations

$$\left[a_r(k), a_s^\dagger(k')\right]_- = \zeta_r \delta_{rs} \delta^3(\boldsymbol{k} - \boldsymbol{k}')\,. \tag{8.56}$$

Of course, in addition, one also obtains

$$\begin{aligned} [a_r(k), a_s(k')]_- &= 0\,, \\ \left[a_r^\dagger(k), a_s^\dagger(k')\right]_- &= 0\,. \end{aligned} \tag{8.57}$$

It follows that the commutator of a_0 with $a_0^\dagger$ has the wrong sign compared to the commutators for scalar fields. This is of course related to the wrong normalization of ϵ_0 shown in Eqs. (8.47) and (8.48). We will discuss its physical implications in §8.6.

□ **Exercise 8.4** Show the equivalence of Eqs. (8.32) and (8.56).

8.6 Physical states

We introduced four polarization vectors in Eq. (8.52). We also found that the creation and annihilation operators associated with the polarization vector ϵ_0 does not have the correct commutation relations. Let us now try to understand what they imply.

As with scalar and fermionic fields, we define the vacuum state by

$$a_r(k)\,|0\rangle = 0 \quad \text{for all } k \text{ and all } r\,, \tag{8.58}$$

and assume that this state is normalized, i.e.,

$$\langle 0\,|0\rangle = 1\,. \tag{8.59}$$

The state containing one photon with momentum k and polarization vector ϵ_r is given by

$$|k,r\rangle \equiv a_r^\dagger(k)\,|0\rangle\ . \tag{8.60}$$

In the case of scalar fields, states defined in a similar way in Eq. (3.39) have positive norm. This is however not guaranteed in this case. Indeed, if we consider a state with polarization vector ϵ_0, we find that

$$\begin{aligned}\langle k,0\,|k',0\rangle &= \left\langle 0\left|a_0(k)a_0^\dagger(k')\right|0\right\rangle \\ &= \left\langle 0\left|\left[a_0(k),a_0^\dagger(k')\right]_-\right|0\right\rangle\,,\end{aligned} \tag{8.61}$$

since the extra term included in the last step vanishes anyway due to the definition of the vacuum state in Eq. (8.58). Using the commutation relation from Eq. (8.56) and recalling that $\zeta_0 = -1$, we get

$$\langle k,0\,|k',0\rangle = -\,\delta^3(\boldsymbol{k}-\boldsymbol{k}')\,. \tag{8.62}$$

This norm is negative, which goes against all we know about quantum mechanics.

The number of states also seem to pose a puzzle. Classical electromagnetic fields have only two independent degrees of freedom, as mentioned earlier. However, in the quantized version, we seem to have four polarization states which are linearly independent. Somehow there must be some redundancy in the definition of these polarization states.

These two problems are related, and in order to resolve them we have to take into account one crucial fact which we swept under the

rug a little while ago. We are not quantizing Maxwell's electrodynamics, but a theory which agrees with it if and only if $\partial_\mu A^\mu = 0$. But we cannot impose $\partial_\mu A^\mu = 0$ on the quantum operator A_μ.

However, Gupta and Bleuler realized that this is not really necessary in order to make a correspondence with the classical theory. All that is needed is that for any two physical states $|\Psi\rangle$ and $|\Psi'\rangle$, the matrix element of $\partial_\mu A^\mu$ between the states vanishes:

$$\langle \Psi' | \partial_\mu A^\mu | \Psi \rangle = 0 \tag{8.63}$$

Let us now define a physical state as any state which is annihilated by the annihilation operator part of $\partial_\mu A^\mu$. We denote this part by $\partial_\mu A^\mu_+$, in keeping with the notation introduced for scalar fields in Eq. (5.44), and write this condition as

$$\partial_\mu A^\mu_+ |\Psi\rangle = 0 . \tag{8.64}$$

Of course, this will also imply

$$\langle \Psi | \partial_\mu A^\mu_- = 0 , \tag{8.65}$$

where A^μ_- is the part of the field operator which contains the creation operator. Since $\partial_\mu A^\mu = \partial_\mu A^\mu_+ + \partial_\mu A^\mu_-$, Eq. (8.63) is automatically satisfied.

Let us now see what this means. From Eq. (8.45), we can write

$$\begin{aligned} \partial_\mu A^\mu_+(x) &= \partial_\mu \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} \epsilon^\mu_r(k) a_r(k) e^{-ik\cdot x} \\ &= -i \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} k_\mu \epsilon^\mu_r(k) a_r(k) e^{-ik\cdot x} \end{aligned} \tag{8.66}$$

This equation is Lorentz invariant, so we can determine the physical content in any frame and gain insight from that. Let us consider a frame in which $\boldsymbol{k}$ is in the z-direction, i.e.,

$$k^\mu = (\omega_k, 0, 0, \omega_k) \Rightarrow k_\mu = (\omega_k, 0, 0, -\omega_k) , \tag{8.67}$$

where we have used Eq. (8.46). In this frame, Eq. (8.66) will take the form

$$\partial_\mu A^\mu_+(x) = -i \int \frac{d^3k\, \omega_k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a_0(k) - a_3(k)\right] e^{-ik\cdot x} , \tag{8.68}$$

using the polarization vectors given in §8.5. Now, Eq. (8.64) will be satisfied if

$$a_0(k)\,|\Psi\rangle = a_3(k)\,|\Psi\rangle\ , \tag{8.69}$$

and Eq. (8.65) will be satisfied if

$$\langle\Psi|\,a_0^\dagger(k) = \langle\Psi|\,a_3^\dagger(k)\,. \tag{8.70}$$

This fact has very interesting consequences on the number of independent degrees of freedom. To see this, let us employ the Feynman-'t Hooft gauge mentioned earlier, i.e., put the gauge parameter ξ to be equal to unity. In this case, using Eq. (8.43), we see that the canonical momenta are given by

$$\Pi^\mu = -\dot{A}^\mu\,. \tag{8.71}$$

We can now construct the total Hamiltonian and use normal ordering to avoid zero-point catastrophes. Using the Fourier decomposition of the fields A^μ from Eq. (8.45), we obtain

$$\begin{aligned} :H: &= \int d^3k\,\omega_k \sum_{r=0}^{3} \zeta_r a_r^\dagger(k) a_r(k) \\ &= \int d^3k\,\omega_k \left[\sum_{r=1}^{3} a_r^\dagger(k) a_r(k) - a_0^\dagger(k) a_0(k)\right]\ . \end{aligned} \tag{8.72}$$

However, if we take the matrix element of the total Hamiltonian operator between any two physical states Ψ and Ψ', the contributions of the $a_3^\dagger a_3$ and the $a_0^\dagger a_0$ terms cancel because of Eqs. (8.69) and (8.70), and we are left with

$$\langle\Psi'\,|\,:H:\,|\,\Psi\rangle = \int d^3k\,\omega_k \left\langle \Psi' \left| \sum_{r=1}^{2} a_r^\dagger(k) a_r(k) \right| \Psi \right\rangle\ . \tag{8.73}$$

Hence, for the matrix element of the Hamiltonian between two physical states, we find that only two polarization states really contribute. And in fact, this is true for all other observables as well. Thus, we seem to have cured the problem of counting of states mentioned earlier. Physical quantities do depend on two independent polarization vectors only. Moreover, none of these two is the scalar polarization state, so we need not bother about the problem of negative norm states.

□ **Exercise 8.5** Among the four states defined in Eq. (8.60), which ones are physical for a photon moving in the z-direction?

□ **Exercise 8.6** Find the total momentum of the free photon field P^μ, normal ordered and for an arbitrary value of the gauge parameter ξ, in terms of the creation and annihilation operators. Show that for physical states it depends only on the two transverse modes of polarization.

8.7 Another look at the propagator

The propagator is not a physically measurable quantity, so it depends on all polarization states. Let us take a closer look at the propagator, trying to separate parts of it that describe different kinds of interactions. We shall work in the Feynman-'t Hooft gauge ($\xi = 1$), in which the propagator reads

$$D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2 + i\varepsilon} . \tag{8.74}$$

Using the completeness relation for the polarization vectors, we can rewrite this as

$$D_{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon} \sum_{r=0}^{3} \zeta_r \epsilon_r^\mu \epsilon_r^{*\nu} . \tag{8.75}$$

It will prove useful to go to a definite frame, so let us go to the previously chosen frame with

$$\begin{aligned} \epsilon_0^\mu &= n^\mu = (1,0,0,0) , \\ \epsilon_3^\mu &= \frac{k^\mu - (k\cdot n)n^\mu}{\sqrt{(k\cdot n)^2 - k^2}} . \end{aligned} \tag{8.76}$$

In this frame the propagator takes the form

$$\begin{aligned} D^{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon} \Bigg[& \sum_{r=1,2} \epsilon_r^\mu(k)\epsilon_r^\nu(k) \\ & + \frac{(k^\mu - (k\cdot n)n^\mu)\,(k^\nu - (k\cdot n)n^\nu)}{(k\cdot n)^2 - k^2} - n^\mu n^\nu \Bigg] . \end{aligned} \tag{8.77}$$

The first term is exclusively for transverse polarizations, and we shall interpret it as the exchange of transverse photons. The last two terms can be rearranged as a sum of two terms,

$$D_C^{\mu\nu}(k) = \frac{n^\mu n^\nu}{(k\cdot n)^2 - k^2} , \tag{8.78}$$

and

$$D_R^{\mu\nu}(k) = \frac{k^\mu k^\nu - (k\cdot n)(n^\mu k^\nu + k^\mu n^\nu)}{k^2[(k\cdot n)^2 - k^2]}\,. \tag{8.79}$$

In our chosen frame, $n^\mu = g^{\mu 0}$ and $k\cdot n = k^0$, so we can write

$$\begin{aligned} D_C^{\mu\nu}(x-x') &= \int \frac{d^4k}{(2\pi)^4} D_C^{\mu\nu}(k) e^{-ik\cdot(x-x')} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{g^{\mu 0}g^{\nu 0}}{\mathbf{k}^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \int \frac{dk^0}{2\pi} e^{-ik^0(x^0-x'^0)} \\ &= \frac{1}{4\pi}\frac{g^{\mu 0}g^{\nu 0}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\delta(x^0-x'^0)\,. \end{aligned} \tag{8.80}$$

□ **Exercise 8.7** Show that

$$\int d^3x\, \frac{e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}}{4\pi r} = \frac{1}{\mathbf{q}^2}\,. \tag{8.81}$$

[Hint: This can be done only as a limiting procedure. For example, you can take the Fourier transform of the function $\frac{\exp(-r/\lambda)}{4\pi r}$ and then take the limit $\lambda \to \infty$. Alternatively, think of the Green's function for Laplace's equation.]

The significance of these different parts can be understood if we consider a scattering process involving the photon as the intermediate particle. Since the interaction with other fields is given by $j^\mu A_\mu$, the matrix element for this process can be written as

$$\int d^4x \int d^4x'\; j_1^\mu(x) D_{\mu\nu}(x-x') j_2^\nu(x')\,, \tag{8.82}$$

where $j_1^\mu(x)$ and $j_2^\nu(x')$ are two current densities interacting via the photon field. The $D_C^{\mu\nu}$ part of the propagator contributes

$$\int d^4x \int d^4x'\, \frac{j_1^0(x) j_2^0(x')}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\delta(x^0-x'^0) \tag{8.83}$$

to this matrix element. This looks like the instantaneous Coulomb interaction between the two charge densities $j_1^0(x)$ and $j_2^0(x')$. Consequently we can interpret $D_C^{\mu\nu}$ as the Coulomb part of the propagator. In order to understand the remainder term $D_R^{\mu\nu}$, we note that the matrix element of Eq. (8.82) can be written in momentum space as

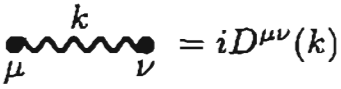
$$\int \frac{d^4k}{(2\pi)^4} j_1^\mu(-k) D_{\mu\nu}(k) j_2^\nu(k)\,. \tag{8.84}$$

Both the currents are conserved currents and therefore satisfy $\partial_\mu j^\mu(x) = 0$. In momentum space, this conservation law reads $k_\mu j^\mu(k) = 0$, again for both currents. Therefore all terms proportional to k_μ or k_ν in $D_{\mu\nu}$ are in fact irrelevant. So $D_R^{\mu\nu}(k)$ does not contribute to the interaction matrix element.

This is also the reason why we can get away with the Feynman-'t Hooft gauge. If we had not set $\xi = 1$, the extra terms in the propagator would have been proportional to $k_\mu k_\nu$, which vanish when the propagator couples to a conserved current.

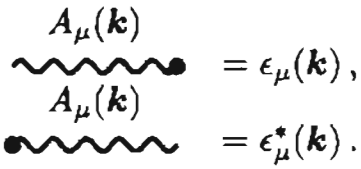
8.8 Feynman rules for photons

We can now summarize the Feynman rules for photons. For an internal photon, the rule is



$$= iD^{\mu\nu}(k) \tag{8.85}$$

irrespective of the direction of the momentum since the photon carries no charge. Sometimes we will put arrows on internal photon lines to indicate the direction of momentum. For external photon lines, the rule is



(8.86)

In calculating cross sections involving external photons, we will need the sum over transverse polarizations only. This is the first term in the square brackets in Eq. (8.77). Using Eq. (8.74), we can write it as

$$\sum_{r=1,2} \epsilon_r^\mu(k)\epsilon_r^\nu(k) = -g^{\mu\nu} - k^2\left[D_C^{\mu\nu}(k) + D_R^{\mu\nu}(k)\right]\,. \tag{8.87}$$

For physical photons, $k^2 = 0$, which implies $k^2 D_C^{\mu\nu}(k) = 0$. However, $k^2 D_R^{\mu\nu}(k)$ need not vanish since $D_R^{\mu\nu}(k)$ has a factor of $1/k^2$. But the photon couples to conserved currents, so $D_R^{\mu\nu}(k)$ has no contribution to physical amplitudes. So we can safely write

$$\sum_{r=1,2} \epsilon_r^\mu(k)\epsilon_r^\nu(k) = -g^{\mu\nu} \tag{8.88}$$

for the polarization sum of physical photons in any scattering amplitude.

Current conservation has played a crucial role in our arguments so far. Any current appearing in Maxwell's equation, Eq. (8.8), must satisfy $\partial_\mu j^\mu = 0$, and the dynamics of the fields constituting the current has to be chosen according to this. We shall now see that there is a simple principle which fixes the interaction between the photon field and fields which act as its sources. These interactions of the photon field are the subject of the next chapter.

Chapter 9

Quantum electrodynamics

9.1 Local gauge invariance

As we know from our discussion on scalar fields in Ch. 3, the Lagrangian of the free complex scalar field is invariant under a global (i.e., space-time independent) phase rotation,

$$\phi \to \phi' = e^{-ieQ\theta}\phi\,. \tag{9.1}$$

This is because in the Lagrangian,

$$\mathscr{L}_\phi = (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - m^2 \phi^\dagger \phi\,, \tag{9.2}$$

a $\phi^\dagger$ appears in any term where ϕ appears. So the phases cancel between ϕ and $\phi^\dagger$. In Eq. (9.1), e is the unit of electric charge, which is taken to be the charge of the proton. And Q is the charge of the particle in this unit.

Such invariance exists in the Lagrangian of the free Dirac field:

$$\mathscr{L}_\psi = \overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi\,. \tag{9.3}$$

Since in both the terms we have both ψ and $\overline{\psi}$, this Lagrangian is invariant under the transformation

$$\psi \to \psi' = e^{-ieQ\theta}\psi\,. \tag{9.4}$$

As in the case of complex scalars, θ is a *global* parameter, i.e., the value of θ is the same at all space-time points. The invariance of the

action under a global transformation implies a conserved current by Noether's theorem, as shown in §2.4. In this case, the current is

$$j^\mu = eQ\overline{\psi}\gamma^\mu\psi\,. \tag{9.5}$$

The conserved charge corresponding to this was shown in Eq. (4.108) to be

$$q = eQ\int d^3x\,\psi^\dagger\psi = eQ\int d^3p\sum_r\left(f_r^\dagger(\boldsymbol{p})f_r(\boldsymbol{p}) - \widehat{f}_r^\dagger(\boldsymbol{p})\widehat{f}_r(\boldsymbol{p})\right), \tag{9.6}$$

where the f_r, $f_r^\dagger$, $\widehat{f}_r$, $\widehat{f}_r^\dagger$ are the creation and annihilation operators for fermions and antifermions. In principle, this could be any charge that comes as both positive and negative. For example it could be electron number minus positron number, or proton number minus antiproton number, or more generally lepton number or baryon number. We shall consider a special case, where it is the electric charge. Any electric charge must couple to the electromagnetic field. There is a remarkable principle called the gauge principle which allows us to determine this coupling.

Suppose we decide to view the phase rotation symmetry as a local symmetry, i.e., have the Lagrangian invariant under the transformations even when θ is a function of space-time. If we are talking about the Lagrangians of Eqs. (9.2) and (9.3), we can see that we cannot do this. Concentrating for the moment on the case of fermions, we find that if we substitute ψ' from Eq. (9.4), the Lagrangian becomes

$$\begin{aligned}\mathscr{L}_{\psi'} &= \overline{\psi}'\,(i\partial\!\!\!/ - m)\,\psi' \\ &= \mathscr{L}_\psi + eQ(\partial_\mu\theta)\overline{\psi}\gamma^\mu\psi\,.\end{aligned} \tag{9.7}$$

As we can see, the fermion Lagrangian fails to remain invariant under a local phase rotation. We can work around this failure by introducing new fields so that the modified Lagrangian has the local symmetry. In the transformed Lagrangian in Eq. (9.7), the extra term on the right side involves $\partial_\mu\theta$, which transforms like a 4-vector. To compensate for this term, we need to introduce another 4-vector in the theory. Let us call it A_μ and write a new Lagrangian:

$$\mathscr{L} = \overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi - eQ\overline{\psi}\gamma^\mu\psi A_\mu\,. \tag{9.8}$$

If we now make a local transformation defined by

$$\psi \to \psi' = e^{-ieQ\theta(x)}\psi\,, \qquad A_\mu \to A'_\mu\,, \tag{9.9}$$

where $\theta(x)$ is a local function on space-time, we can calculate what A'_μ should be so that the Lagrangian of Eq. (9.8) remains invariant under this transformation. The transformed Lagrangian is

$$\begin{aligned}\mathscr{L}' &= \overline{\psi}'\left(i\partial\!\!\!/ - m\right)\psi' - eQ\overline{\psi}'\gamma^\mu\psi' A'_\mu \\ &= \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - eQ(A'_\mu - \partial_\mu\theta)\overline{\psi}\gamma^\mu\psi \end{aligned} \tag{9.10}$$

This will be the same Lagrangian as the one of Eq. (9.8) if

$$A'_\mu = A_\mu + \partial_\mu\theta \tag{9.11}$$

But this transformation is familiar to us. This is the same local transformation which keeps the free Lagrangian of the electromagnetic field invariant (see Eq. (8.12)). So let us identify this field A_μ with the photon field and introduce in the Lagrangian the term for the free A_μ field, Eq. (8.11). Adding the free term, we can write down the full Lagrangian for the fields ψ and A_μ,

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\overline{\psi}\gamma^\mu\psi A_\mu\,. \tag{9.12}$$

If we look at only the terms containing photon field, we find exactly the terms given in Eq. (8.10), since the conserved current is given by

$$j^\mu = eQ\overline{\psi}\gamma^\mu\psi \tag{9.13}$$

in this case, which was derived in Eq. (4.92). This confirms our guess that A_μ can be thought of as the electromagnetic field. The field theory with the Lagrangian of Eq. (9.12) is called *quantum electrodynamics*, or QED for short.

Let us summarize what we have done. We had a global symmetry in the free Lagrangian of the fermion field. We made this symmetry local and demanded that the action remain invariant under this local symmetry. This required the introduction of the photon field A_μ through a $\overline{\psi}\gamma^\mu A_\mu\psi$ term. This procedure is known as *gauging* the global symmetry, and the hypothesis that the action can be made invariant by introducing new local fields is called the *gauge principle*. The field A_μ is called the gauge boson corresponding to the local symmetry. Classically (which is how we implemented the gauge principle) the coefficient of this term, known as the coupling constant, is arbitrary up to the normalization of fields. However, it so happens

that the electric charge of all fermions are integral multiples of some fundamental charge. The problem of finding a suitable explanation for charge quantization remains open to this day.

It may seem from our presentation that the procedure works specifically for fermion fields. But that is not so. It works for any other field with similar global symmetry. To appreciate that, let us write Eq. (9.12) in the following form:

$$\mathscr{L} = \overline{\psi}\,(i\gamma^\mu D_\mu - m)\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \tag{9.14}$$

where D_μ is called the *gauge covariant derivative*:

$$D_\mu = \partial_\mu + ieQA_\mu\,. \tag{9.15}$$

For any complex field ϕ transforming as $\phi \to e^{-ieQ\theta}\phi$, its gauge covariant derivative transforms as $D_\mu\phi \to e^{-ieQ\theta}D_\mu\phi$ if we also implement the gauge transformation $A_\mu \to A_\mu + \partial_\mu\theta$. Thus all terms that were invariant under the global phase rotation remain invariant under the local phase rotation provided we make the *minimal substitution* $\partial_\mu \to D_\mu$. The Lagrangian of Eq. (9.12) is constructed by making the minimal substitution in the free Lagrangian and adding in the Lagrangian for the free electromagnetic field. This prescription can also be used for constructing the interactions of a charged scalar field with the electromagnetic field. This will give us the Lagrangian of *scalar electrodynamics*, which is

$$\mathscr{L} = (D^\mu\phi)^\dagger(D_\mu\phi) - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.16}$$

Of course, any non-derivative interaction term such as $V(\phi^\dagger\phi)$ remains untouched by minimal substitution.

□ **Exercise 9.1** Show that the Lagrangian of Eq. (9.16) is invariant under local phase transformations, where ϕ transforms by Eq. (9.1) with θ being a function of space-time, and A_μ transforms according to Eq. (9.11).

□ **Exercise 9.2** Show that, if ψ transform by Eq. (9.4) with a space-time dependent θ, and A_μ transforms by Eq. (9.11), the transformation of $D_\mu\psi$ is the same as that of ψ, i.e.,

$$D'_\mu\psi' = e^{-ieQ\theta}D_\mu\psi\,, \tag{9.17}$$

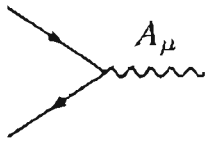
where $D'_\mu = \partial_\mu + ieQA'_\mu$.

9.2 Interaction Hamiltonian

Quantum electrodynamics, the theory of photons and fermions, is described by the Lagrangian given in Eq. (9.12). In addition to the free terms for the fields ψ and A_μ, it contains an interaction term,

$$\mathscr{L}_{\text{int}} = -eQ\bar{\psi}\gamma^\mu\psi A_\mu \,, \tag{9.18}$$

which represents an interaction between the two kinds of fields. We can write down the Feynman rule for the vertex by stripping $\mathscr{L}_{\text{int}}$ of all the field operators and multiplying by i:



$$: \quad -ieQ\,\gamma^\mu \,. \tag{9.19}$$

Here and in general, we will ignore time directions when writing the Feynman rule for vertices. The external arrows of the vertex will be positioned as appropriate to the process being considered.

Let us consider the specific case of the electron field. The electron charge is $-e$, which means $Q = -1$. So for the interaction of the electron field with photons, the normal-ordered interaction Hamiltonian is given by

$$:\mathscr{H}_I: \; = -e:\bar{\psi}\gamma^\mu\psi A_\mu: \,. \tag{9.20}$$

To understand the nature of this interaction, let us use the notation introduced in Eq. (5.44), where the creation and the annihilation operator parts of the field operator where shown separately. If we do a similar thing for the electron and the photon fields here, we will obtain eight different terms. This will give rise to eight different kind of events occurring at an interaction vertex. All eight are shown in Fig. 9.1. Let us try to describe these terms in words and pictures.

First, consider the term $:\bar{\psi}_+\gamma_\mu\psi_+A^\mu_+:$. The operator ψ_+ annihilates an electron, $\bar{\psi}_+$ annihilates a positron, and A^μ_+ annihilates a photon. Thus, this term describes a photon being annihilated along with an electron-positron pair, as in Fig. 9.1a. Obviously, this cannot be a physical process since the final state has no particles to carry the energies of the particles in the initial state.

Let us consider another term, e.g., $:\bar{\psi}_-\gamma_\mu\psi_+A^\mu_-:$. In this case, an electron is annihilated, and an electron and a photon are created, as

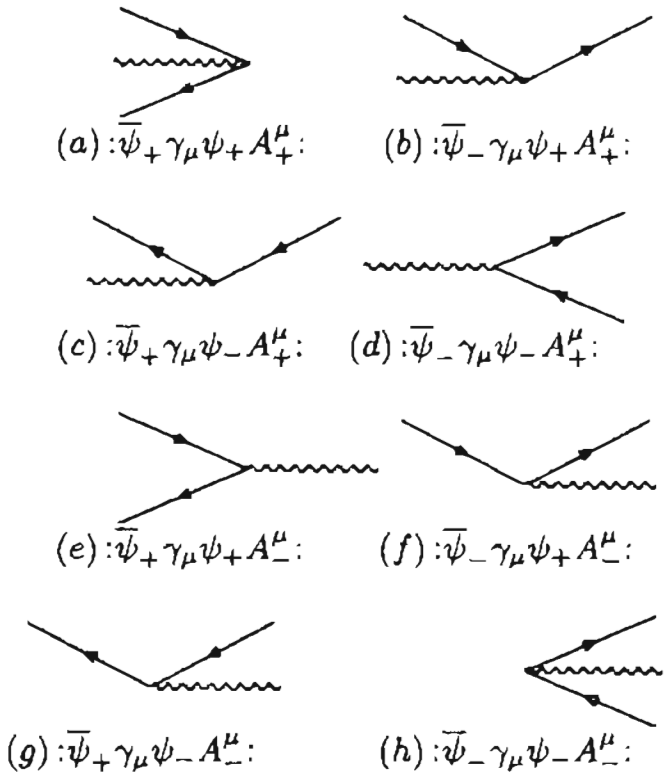


Figure 9.1: The different events that can occur at a vertex with the QED Hamiltonian.

shown in Fig. 9.1f. Carrying on in a similar way, we can summarize the possibilities as follows. At each vertex, there will be two fermion lines and one photon line. The fermion line corresponding to the operator ψ can be either annihilation of an electron or creation of a positron. The other fermion line corresponds to the operator $\overline{\psi}$, which can either create an electron or annihilate a positron. The photon line can correspond to the creation or the annihilation of a photon. This gives eight combinations, which are shown in Fig. 9.1.

For one of these eight cases, we have already shown that the event occurring at a vertex cannot occur as a physical process. This is the case for all eight vertices. Let us denote the 4-momenta of the two fermion lines by p and p', and that of the photon line by k. Then for all these vertices momentum conservation will yield an equation of the form

$$p' = \pm p \pm k \,, \tag{9.21}$$

for some combination of signs on the right side. Squaring both sides,

we obtain

$$p'^2 = p^2 + k^2 \pm 2p \cdot k\,. \tag{9.22}$$

If all of these particles are on-shell, $p^2 = p'^2 = m^2$ and $k^2 = 0$, where m is the mass of the electron. Thus we obtain

$$p \cdot k = 0\,. \tag{9.23}$$

Since this is a Lorentz-invariant equation, it is valid in any inertial frame. In particular it is valid in the frame where one of the fermions is at rest, $p = (m, 0, 0, 0)$. Then in that frame, k is spacelike, $k = (0, \boldsymbol{k})$. But since $k^2 = 0$, it follows that $\boldsymbol{k} = 0$, i.e., the photon does not exist, and therefore the process does not exist. Of course, the vertex can be *inside* a bigger diagram describing a physical process, because virtual particles inside a diagram do not have to obey $p^2 = p'^2 = m^2$, or $k^2 = 0$.

□ **Exercise 9.3** In an arbitrary frame, show that Eq. (9.23) implies

$$\cos\theta = \frac{E\omega}{\mathbf{p}\,\mathbf{k}}\,, \tag{9.24}$$

where θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{k}$. Argue that this cannot be satisfied for physical particles.

9.3 Lowest order processes

We have already seen that it is not possible to obtain a physical process with only one QED vertex. In other words, the first order term in the S-matrix expansion has no matrix element between physical states. So in order to look at some physical processes we have to go at least to the second order in the S-matrix expansion, i.e., we need to consider Feynman diagrams with two vertices. Let us see what the possibilities are in that case.

As we mentioned, not all the lines occurring at any single vertex can be external lines. So, let us first consider the possibility that only one line at each vertex is internal. If the internal line happens to be a photon, we will have only the four fermions as external lines. Since charge conservation has to be maintained, there can be only

the following processes in this category:

$$\begin{aligned} e^- + e^- &\to e^- + e^- , \\ e^- + e^+ &\to e^- + e^+ , \\ e^+ + e^+ &\to e^+ + e^+ . \end{aligned} \tag{9.25}$$

These processes will be discussed in detail in §9.4 and §9.5.

Of course, there are other charged particles in nature. The photon field couples to them just as well. If we consider charged fermions, the Feynman rule for the interaction will be given by Eq. (9.19). So in the lowest order scattering process, some external lines may be some other charged fermion. We consider one such process, $e^- + e^+ \to \mu^- + \mu^+$, in §9.6.

If there is only one internal line, it can also be a fermion line. In this case, two of the external lines will be fermion lines and the other two, photon lines. So we can have only the following processes:

$$\begin{aligned} e^- + \gamma &\to e^- + \gamma , \\ e^+ + \gamma &\to e^+ + \gamma , \\ e^- + e^+ &\to \gamma + \gamma , \\ \gamma + \gamma &\to e^- + e^+ \end{aligned} \tag{9.26}$$

Here the first one is electron-photon scattering, often called Compton scattering. This will be discussed in §9.8. The second one is positron-photon scattering. The last two are inelastic processes where the initial and final particles are different. The process $e^- + e^+ \to \gamma + \gamma$ is called pair annihilation. The reverse process is called pair creation.

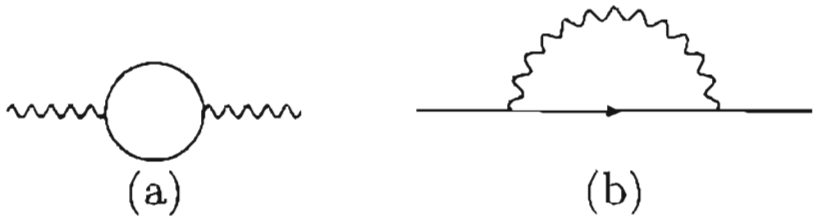


Figure 9.2: Second order diagrams with two internal lines.

Let us now consider the diagrams with two internal lines. Here there are two possibilities. First, both internal lines could be fermion

lines, in which case we get the diagram of Fig. 9.2a. On the other hand, one of the internal lines can be a photon, in which case we get Fig. 9.2b. These are not diagrams for scattering, since the initial state does not have two particles. These cannot be decay processes either, since one particle cannot decay to one particle in the final state. These are called *self-energy diagrams*. The first one, where the external lines are photon lines, is called the self-energy diagram for the photon, or sometimes vacuum polarization. The second one is the self-energy diagram for the fermion. The interpretation of these diagrams, along with the evaluation of these amplitudes, will be discussed in Ch. 12.

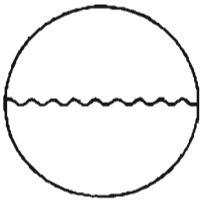


Figure 9.3: Second order diagram with three internal lines.

Finally, let us consider the possibility that all lines appearing in both vertices are internal lines. This diagram has been shown in Fig. 9.3. This represents a vacuum to vacuum amplitude since there is no line left to be an external line. Since the vacuum of QED is stable, we shall henceforth ignore such diagrams for reasons discussed in §6.1.

□ **Exercise 9.4** The second order term in the S-matrix expansion is:

$$S^{(2)} = -\frac{e^2}{2}\int d^4x_1 \int d^4x_2\, \mathscr{T}\left[:(\bar{\psi}\gamma^\mu\psi A_\mu)_{x_1}:\ :(\bar{\psi}\gamma^\nu\psi A_\nu)_{x_2}:\right]. \tag{9.27}$$

Use Wick's theorem to write the time-ordered product appearing here in terms of normal-ordered products. Identify all the processes discussed above as different terms in the Wick expansion.

9.4 Electron-electron scattering

The first example of a QED scattering process we shall consider is electron-electron scattering:

$$e^-(p_1) + e^-(p_2) \to e^-(p_1') + e^-(p_2'), \tag{9.28}$$

where p_1, p_2, etc. are the 4-momenta of the different particles. This is one of the processes which occur at the second order in the S-matrix expansion. There are two Feynman diagrams corresponding to this process, shown in Fig. 9.4. The two diagrams come from the same S-matrix term, as in Fig. 6.4.

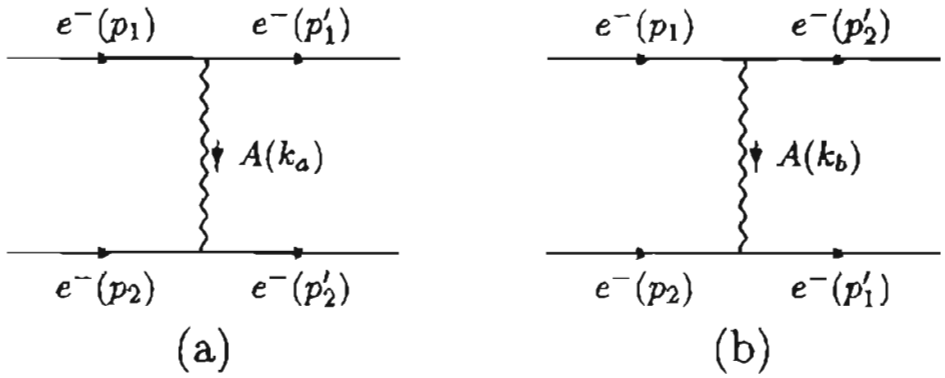


Figure 9.4: Lowest order diagrams for electron-electron scattering in QED.

The Feynman amplitude

We can write down the Feynman amplitude for these diagrams using the Feynman rules of QED. For the diagram marked (a), we obtain

$$i\mathcal{M}_a = [\overline{u}(p_1')(ie\gamma_\mu)u(p_1)]iD^{\mu\nu}(k_a)[\overline{u}(p_2')(ie\gamma_\nu)u(p_2)]\,, \quad (9.29)$$

where $k_a = p_1 - p_1' = p_2' - p_2$, which comes from the conservation of 4-momentum at the two vertices. For the other diagram, we obtain

$$i\mathcal{M}_b = [\overline{u}(p_1')(ie\gamma_\mu)u(p_2)]iD^{\mu\nu}(k_b)[\overline{u}(p_2')(ie\gamma_\nu)u(p_1)]\,, \quad (9.30)$$

where $k_b = p_1 - p_2' = p_1' - p_2$. Since the two diagrams are related by an exchange of the final particles, we should add them with a relative minus sign between the amplitudes of the two diagrams, as we found at the end of §6.4.

Now we have to substitute the expression for the photon propagator. In order to show the gauge-independence of the Feynman amplitude, let us use the expression in Eq. (8.40),

$$D_{\mu\nu}(k) = -\frac{1}{k^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]. \quad (9.31)$$

From the definition of k_a above, we have

$$k_a^\mu[\overline{u}(p_1')\gamma_\mu u(p_1)] = [\overline{u}(p_1')\,(\not{p}_1 - \not{p}_1')\,u(p_1)]\,. \tag{9.32}$$

Now we use the relations $\not{p}u(p) = mu(p)$ and $\overline{u}(p)\not{p} = m\overline{u}(p)$, which were given in Eqs. (4.46) and (4.48), and find that this expression vanishes. As a result, the terms involving k_a^μ in the propagator do not contribute to the amplitude. Similarly, for the other diagram the terms involving k_b^μ do not contribute. Since these are the terms which contain the gauge-fixing parameter ξ, we have shown that the amplitude does not depend on ξ. We are thus left with only the $g_{\mu\nu}$-term in the propagators, and this gives the following expression for the total amplitude:

$$\begin{aligned}\mathcal{M} &\equiv \mathcal{M}_a - \mathcal{M}_b \\ &= e^2\left\{\frac{[\overline{u}_1'\gamma_\mu u_1][\overline{u}_2'\gamma^\mu u_2]}{(p_1-p_1')^2} - \frac{[\overline{u}_1'\gamma_\mu u_2][\overline{u}_2'\gamma^\mu u_1]}{(p_1-p_2')^2}\right\},\end{aligned} \tag{9.33}$$

where we have written $\overline{u}_1'$ for $\overline{u}(p_1')$ etc. This expression is clearly antisymmetric if we interchange the two electrons in the final state. Since the denominator in the second term can also be written as $(p_1' - p_2)^2$, it is straightforward to see that the expression is also antisymmetric under the interchange of the two electrons in the initial state. This is of course how it should be, since the electrons are fermions.

Our aim is to calculate the scattering cross section. For this, we will need the absolute square of the Feynman amplitude. If we do not care about the spin of the electrons in either the initial or the final state, we will need to sum this over the final spins and average over the initial spins. The techniques for doing this have already been described in §7.2.1. In this case, averaging over the initial spins would involve a division by 4, since each electron can have two spin states. Thus we obtain

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 \\ &= \frac{e^4}{4}\sum_{\text{spin}}\left\{\frac{[\overline{u}_1'\gamma_\mu u_1][\overline{u}_2'\gamma^\mu u_2]}{(p_1-p_1')^2} - \frac{[\overline{u}_1'\gamma_\mu u_2][\overline{u}_2'\gamma^\mu u_1]}{(p_1-p_2')^2}\right\} \\ &\qquad \times\left\{\frac{[\overline{u}_1\gamma_\nu u_1'][\overline{u}_2\gamma^\nu u_2']}{(p_1-p_1')^2} - \frac{[\overline{u}_2\gamma_\nu u_1'][\overline{u}_1\gamma^\nu u_2']}{(p_1-p_2')^2}\right\}.\end{aligned} \tag{9.34}$$

The sum over spins can be turned into traces, as we discussed in various examples of Ch. 7. Here we obtain

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4}\left\{\frac{T_{11}}{(p_1-p_1')^4} + \frac{T_{22}}{(p_1-p_2')^4} - \frac{T_{12}+T_{21}}{(p_1-p_1')^2(p_1-p_2')^2}\right\}, \tag{9.35}$$

where

$$\begin{aligned}
T_{11} &= \mathrm{Tr}\,[(\not p_1 + m)\gamma_\nu(\not p_1' + m)\gamma_\mu]\,\mathrm{Tr}\,[(\not p_2 + m)\gamma^\nu(\not p_2' + m)\gamma^\mu]\,,\\
T_{22} &= \mathrm{Tr}\,[(\not p_2 + m)\gamma_\nu(\not p_1' + m)\gamma_\mu]\,\mathrm{Tr}\,[(\not p_1 + m)\gamma^\nu(\not p_2' + m)\gamma^\mu]\,,\\
T_{12} &= \mathrm{Tr}\,[(\not p_1 + m)\gamma^\nu(\not p_2' + m)\gamma^\mu(\not p_2 + m)\gamma_\nu(\not p_1' + m)\gamma_\mu]\,,\\
T_{21} &= \mathrm{Tr}\,[(\not p_1 + m)\gamma_\nu(\not p_1' + m)\gamma_\mu(\not p_2 + m)\gamma^\nu(\not p_2' + m)\gamma^\mu]\,.
\end{aligned} \tag{9.36}$$

We will now calculate these traces.

Calculation of the traces

Evaluation of the traces is straightforward, using the trace formulas. For example, for any two vectors a and b,

$$\mathrm{Tr}\,[(\not a + m)\gamma_\nu(\not b + m)\gamma_\mu] = 4\left[a_\nu b_\mu + b_\nu a_\mu - g_{\mu\nu}(a\cdot b - m^2)\right], \tag{9.37}$$

which takes care of the traces occurring in T_{11} and T_{22}. Thus we obtain

$$\begin{aligned}
T_{11} &= 16[p_{1\nu}p_{1\mu}' + p_{1\mu}p_{1\nu}' - g_{\mu\nu}(p_1\cdot p_1' - m^2)]\\
&\qquad\qquad \times [p_2^\nu p_2'^\mu + p_2^\mu p_2'^\nu - g^{\mu\nu}(p_2\cdot p_2' - m^2)]\\
&= 32[(p_1\cdot p_2)^2 + (p_1\cdot p_2')^2 + 2m^2(m^2 - p_1\cdot p_1')]\,,
\end{aligned} \tag{9.38}$$

where in the last step we have used

$$p_1\cdot p_2 = p_1'\cdot p_2' \tag{9.39}$$

and other similar equations, which follow from momentum conservation, $p_1 + p_2 = p_1' + p_2'$. Similarly, the second term can be obtained by interchanging p_1' and p_2', which gives

$$T_{22} = 32[(p_1\cdot p_2)^2 + (p_1\cdot p_1')^2 + 2m^2(m^2 - p_1\cdot p_2')]\,. \tag{9.40}$$

Let us now look at the other two terms. Since an odd number of γ-matrices have vanishing trace, we can write

$$\begin{aligned} T_{12} = & \mathrm{Tr}\left[\not{p}_1\gamma^\nu\not{p}'_2\gamma^\mu\not{p}_2\gamma_\nu\not{p}'_1\gamma_\mu\right] \\ & +m^2\mathrm{Tr}\left[\not{p}_1\gamma^\nu\not{p}'_2\gamma^\mu\gamma_\nu\gamma_\mu + \not{p}_1\gamma^\nu\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu + \not{p}_1\gamma^\nu\gamma^\mu\gamma_\nu\not{p}'_1\gamma_\mu\right. \\ & \quad \left. +\gamma^\nu\not{p}'_2\gamma^\mu\not{p}_2\gamma_\nu\gamma_\mu + \gamma^\nu\not{p}'_2\gamma^\mu\gamma_\nu\not{p}'_1\gamma_\mu + \gamma^\nu\gamma^\mu\not{p}_2\gamma_\nu\not{p}'_1\gamma_\mu\right] \\ & +m^4\mathrm{Tr}\left[\gamma^\nu\gamma^\mu\gamma_\nu\gamma_\mu\right]. \end{aligned} \tag{9.41}$$

Using the contraction formulas of Eq. (4.42), this can be rewritten as

$$\begin{aligned} T_{12} = & -2\ \mathrm{Tr}\left[\not{p}_1\not{p}_2\gamma^\mu\not{p}'_2\not{p}'_1\gamma_\mu\right] \\ & +4m^2\ \mathrm{Tr}\left[\not{p}_1\not{p}'_2 + \not{p}_1\not{p}_2 + \not{p}_1\not{p}'_1 + \not{p}_2\not{p}'_2 + \not{p}'_1\not{p}'_2 + \not{p}'_1\not{p}_2\right] \\ & -2m^4\ \mathrm{Tr}\left[\gamma^\mu\gamma_\mu\right] \\ = & -32p_1\cdot p_2\, p'_2\cdot p'_1 + 32m^2[p_1\cdot p'_2 + p_1\cdot p_2 + p_1\cdot p'_1] - 32m^4\,. \end{aligned} \tag{9.42}$$

Note that this contribution is unaffected under the interchange $p'_1 \to p'_2$, so T_{21} must be the same as this. Thus we obtain

$$T_{12} + T_{21} = 64\left[-(p_1\cdot p_2)^2 + 2m^2 p_1\cdot p_2\right], \tag{9.43}$$

using Eq. (9.39) and similar equations to simplify the expression.

Differential cross-section in the CM frame

We now need to insert these expressions into the general formula for elastic cross-sections in the CM frame, Eq. (7.107):

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|\mathscr{M}|^2}, \tag{9.44}$$

We have expressed the traces involved in $\overline{|\mathscr{M}|^2}$ in terms of the dot products of various momentum vectors. We can express the denominators of Eq. (9.35) in terms of dot products as well:

$$\begin{aligned} (p_1 - p'_1)^2 &= p_1^2 + {p'_1}^2 - 2p_1\cdot p'_1 = 2(m^2 - p_1\cdot p'_1)\,, \\ (p_1 - p'_2)^2 &= 2(m^2 - p_1\cdot p'_2)\,. \end{aligned} \tag{9.45}$$

All these dot products can be expressed in terms of the Mandelstam variables, of which only two are independent. Here we choose not to use the Mandelstam variables, but the scattering angle θ, as shown in Fig. 7.1, as one of the parameters. The energy of any of the four external particles will be the other parameter. Note that each particle has the same energy E in the CM frame. We then obtain, for the dot products appearing in $\overline{|\mathcal{M}|^2}$, the following expressions:

$$\begin{aligned} p_1 \cdot p_2 &= E^2 + \mathbf{p}^2 , \\ p_1 \cdot p_1' &= E^2 - \mathbf{p}^2 \cos\theta , \\ p_1 \cdot p_2' &= E^2 + \mathbf{p}^2 \cos\theta , \end{aligned} \tag{9.46}$$

where $\mathbf{p}^2 = E^2 - m^2$. This gives $(p_1 - p_1')^2 = -2\mathbf{p}^2(1 - \cos\theta)$ etc. Collecting all the terms then, we obtain

$$\begin{aligned} \overline{|\mathcal{M}|^2} = \frac{2e^4}{\mathbf{p}^4} \Bigg\{ & \frac{(E^2 + \mathbf{p}^2)^2 + (E^2 + \mathbf{p}^2\cos\theta)^2 - 2m^2\mathbf{p}^2(1 - \cos\theta)}{(1 - \cos\theta)^2} \\ & + \frac{(E^2 + \mathbf{p}^2)^2 + (E^2 - \mathbf{p}^2\cos\theta)^2 - 2m^2\mathbf{p}^2(1 + \cos\theta)}{(1 + \cos\theta)^2} \\ & + 2\frac{(E^2 + \mathbf{p}^2)^2 - 2m^2(E^2 + \mathbf{p}^2)}{\sin^2\theta} \Bigg\} . \end{aligned} \tag{9.47}$$

The differential cross section in this frame is then given by Eq. (9.44), with $s = 4E^2$. The total cross section is obtained by integrating over the angular variables. But since the final particles are identical, only half the phase space is available, which means that θ should be integrated from 0 to $\pi/2$.

□ **Exercise 9.5** Argue that the cross section for the positron-positron scattering should be the same.

□ **Exercise 9.6** Show that in the ultra-relativistic limit $E \gg m$, the differential cross-section becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2} \left\{ \frac{1}{\sin^4\frac{\theta}{2}} + \frac{1}{\cos^4\frac{\theta}{2}} + 1 \right\} , \tag{9.48}$$

where $\alpha = e^2/4\pi$ is the fine-structure constant.

9.5 Electron-positron scattering

Next we consider electron-positron scattering,

$$e^-(p_1) + e^+(p_2) \to e^-(p_1') + e^+(p_2') \,. \tag{9.49}$$

This process also occurs at the second order in the S-matrix expansion. There are two diagrams corresponding to this process, shown in Fig. 9.5.

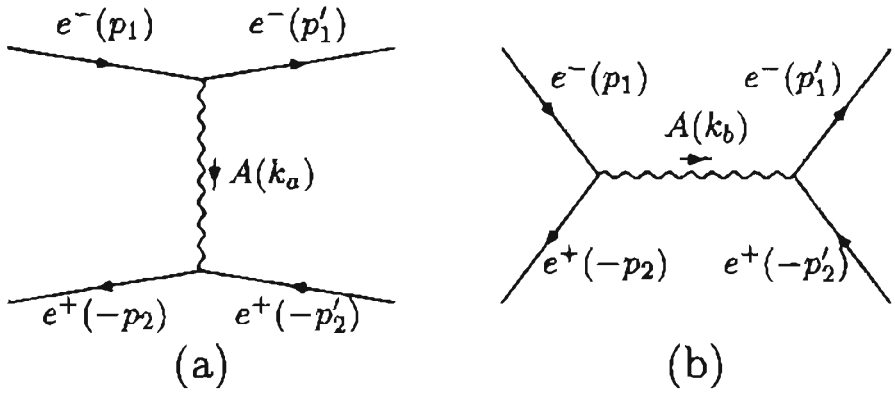


Figure 9.5: Lowest order diagrams for electron-positron scattering in QED.

Just as for the e^-e^- scattering, we shall use the Feynman rules for QED to write down the Feynman amplitudes for these diagrams. The amplitude for the diagram marked (a) is

$$i\mathscr{M}_a = [\overline{u}(p_1')(ie\gamma_\mu)u(p_1)]iD^{\mu\nu}(k_a)[\overline{v}(p_2)(ie\gamma_\nu)v(p_2')] \,. \tag{9.50}$$

Here $k_a = p_1 - p_1' = p_2' - p_2$, since momentum is conserved at each vertex. For diagram (b), we obtain

$$i\mathscr{M}_b = [\overline{v}(p_2)(ie\gamma_\mu)u(p_1)]iD^{\mu\nu}(k_b)[\overline{u}(p_1')(ie\gamma_\nu)v(p_2')] \,, \tag{9.51}$$

where $k_b = p_1 + p_2 = p_1' + p_2'$. The two diagrams are related by a fermion exchange — the initial e^+ is exchanged with the final e^- to get diagram (b) from diagram (a). Therefore the two amplitudes should be added with a relative minus sign between them.

As before, the $k_\mu k_\nu$ terms in the propagator do not contribute because the spinors satisfy Eqs. (4.46) and (4.48), and we are left with the following expression for the total amplitude:

$$\begin{aligned} \mathscr{M} &\equiv \mathscr{M}_a - \mathscr{M}_b \\ &= e^2 \left\{ \frac{[\overline{u}_1'\gamma_\mu u_1][\overline{v}_2\gamma^\mu v_2']}{(p_1 - p_1')^2} - \frac{[\overline{v}_2\gamma_\mu u_1][\overline{u}_1'\gamma^\mu v_2']}{(p_1 + p_2)^2} \right\} . \end{aligned} \tag{9.52}$$

We shall consider the cross section of unpolarized particles, so we sum over the final spins and average over the initial spins. We get

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 \\ &= \frac{e^4}{4}\left|\frac{[\overline{u}_1'\gamma_\mu u_1][\overline{v}_2\gamma^\mu v_2']}{(p_1-p_1')^2} - \frac{[\overline{v}_2\gamma_\mu u_1][\overline{u}_1'\gamma^\mu v_2']}{(p_1+p_2)^2}\right|^2 \\ &= \frac{e^4}{4}\left\{\frac{T_{11}}{(p_1-p_1')^4} + \frac{T_{22}}{(p_1+p_2)^4} - \frac{T_{12}+T_{21}}{(p_1-p_1')^2(p_1+p_2)^2}\right\}.\end{aligned} \tag{9.53}$$

Here again, we can use the techniques of Ch. 7 to write $\overline{|\mathcal{M}|^2}$ in terms of various traces,

$$\begin{aligned}T_{11} &= \mathrm{Tr}\,[(\not p_1+m)\gamma_\nu(\not p_1'+m)\gamma_\mu]\,\mathrm{Tr}\,[(\not p_2'-m)\gamma^\nu(\not p_2-m)\gamma^\mu]\,, \\ T_{22} &= \mathrm{Tr}\,[(\not p_1+m)\gamma_\nu(\not p_2-m)\gamma_\mu]\,\mathrm{Tr}\,[(\not p_2'-m)\gamma^\nu(\not p_1'+m)\gamma^\mu]\,, \\ T_{12} &= \mathrm{Tr}\,[(\not p_1+m)\gamma_\nu(\not p_2-m)\gamma^\mu(\not p_2'-m)\gamma^\nu(\not p_1'+m)\gamma_\mu]\,, \\ T_{21} &= \mathrm{Tr}\,[(\not p_1+m)\gamma_\nu(\not p_1'+m)\gamma^\mu(\not p_2'-m)\gamma^\nu(\not p_2-m)\gamma_\mu]\,.\end{aligned} \tag{9.54}$$

We have seen how to calculate traces like these in §9.4. The results can be obtained directly by interchanging p_2 and $-p_2'$ in Eq. (9.36):

$$\begin{aligned}T_{11} &= 32[(p_1\cdot p_2)^2+(p_1\cdot p_2')^2+2m^2(m^2-p_1\cdot p_1')]\,, \\ T_{22} &= 32[(p_1\cdot p_1')^2+(p_1\cdot p_2')^2+2m^2(m^2+p_1\cdot p_2)]\,, \\ T_{12} &= -32(p_1\cdot p_2')^2-32m^2[p_1\cdot p_2'+p_1\cdot p_2-p_1\cdot p_1']-32m^4\,, \\ T_{21} &= T_{12}\,.\end{aligned} \tag{9.55}$$

We note that T_{22} can be derived from T_{11} by exchanging p_1' with $-p_2$. Similarly, the equality of T_{12} and T_{21} follows from the fact that T_{12} is invariant under the exchange of p_1' with $-p_2$. Of course, these traces can be derived by direct calculations as well.

As before, we will express the dot products of the momenta in terms of the scattering angle θ and the energy E of each particle in the CM frame. In this frame, these will be the same as in Eq. (9.46),

and we can write

$$\overline{|\mathscr{M}|^2} = 2e^4 \Bigg\{ \frac{(E^2+\mathbf{p}^2)^2 + (E^2+\mathbf{p}^2\cos\theta)^2 - 2m^2\mathbf{p}^2(1-\cos\theta)}{\mathbf{p}^4(1-\cos\theta)^2} + \frac{E^4+\mathbf{p}^4\cos^2\theta+2m^2E^2}{2E^4} - \frac{(E^2+\mathbf{p}^2\cos\theta)^2+2m^2(E^2+\mathbf{p}^2\cos\theta)}{E^2\mathbf{p}^2(1-\cos\theta)} \Bigg\}. \quad (9.56)$$

Using $s = 4E^2$ in this frame, we can calculate the differential cross section from Eq. (9.44). Using $\alpha = e^2/4\pi$, we can write this in the high energy limit $E \gg m$ as

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2}\left(\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}} + \frac{1+\cos^2\theta}{2} - \frac{2\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} \right). \quad (9.57)$$

9.6 $e^-e^+ \to \mu^-\mu^+$

Earlier we discussed the elastic scattering process for the electron-positron pair. If the energy of the incoming electron-positron pair is high enough, the scattering need not be elastic. In the final state, we can obtain a particle-antiparticle pair corresponding to a higher mass. Among the known elementary charged particles, the electron is the lightest, and the next lightest is the muon. Here, we calculate the cross-section for muon-antimuon production, although the analysis can be used with minor modification for the production of any fermion-antifermion pair.

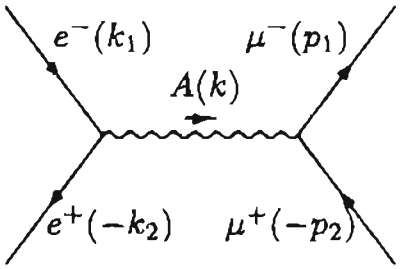


Figure 9.6: Lowest order diagrams for the process $e^-e^+ \to \mu^-\mu^+$ in QED.

There is only one diagram at the lowest order, shown in Fig. 9.6. The muon has the same charge as the electron, and so its vertex with the photon would also be the same as that for the electron. The Feynman amplitude can then be written as

$$\mathscr{M} = \frac{e^2}{s}\left[\bar{v}(k_2)\gamma^\lambda u(k_1)\right]\left[\bar{u}(p_1)\gamma_\lambda v(p_2)\right], \tag{9.58}$$

where s is the Mandelstam variable, which comes from the photon propagator. We have denoted the initial state momenta by k_1 and k_2, and the final state momenta by p_1 and p_2.

In calculating the cross-section, there is no point in keeping the electron mass, since the energies involved must be higher than the muon mass. We will however keep the muon mass m_μ. Then

$$\begin{aligned}
\overline{|\mathscr{M}|^2} &= \frac{1}{4}\sum_{\text{spin}} |\mathscr{M}|^2 \\
&= \frac{e^4}{4s^2}\,\mathrm{Tr}\left[\not{k}_2\gamma^\lambda\not{k}_1\gamma^\rho\right]\mathrm{Tr}\left[(\not{p}_1 + m_\mu)\gamma_\lambda(\not{p}_2 - m_\mu)\gamma_\rho\right] \\
&= \frac{4e^4}{s^2}\left[k_1^\lambda k_2^\rho + k_1^\rho k_2^\lambda - g^{\lambda\rho}k_1\cdot k_2\right] \\
&\qquad\qquad \times\left[p_{1\lambda}p_{2\rho} + p_{1\rho}p_{2\lambda} - g_{\lambda\rho}(p_1\cdot p_2 + m_\mu^2)\right] \\
&= \frac{8e^4}{s^2}\left[(k_1\cdot p_1)^2 + (k_1\cdot p_2)^2 + m_\mu^2 k_1\cdot k_2\right].
\end{aligned} \tag{9.59}$$

We have used kinematical identities like $k_1\cdot p_1 = k_2\cdot p_2$ in writing the last step.

In the CM frame, if θ is the angle between the incoming electron and the outgoing muon, i.e., between $\boldsymbol{k}_1$ and $\boldsymbol{p}_1$, we obtain

$$\begin{aligned}
k_1\cdot p_1 &= E(E - \mathbf{p}\cos\theta), \\
k_1\cdot p_2 &= E(E + \mathbf{p}\cos\theta), \\
k_1\cdot k_2 &= 2E^2.
\end{aligned} \tag{9.60}$$

Here E is the CM energy of the incoming electron, i.e., $s = 4E^2$, and $\mathbf{p} = |\boldsymbol{p}_1| = |\boldsymbol{p}_2| = \frac{1}{2}\sqrt{s - 4m_\mu^2}$. Thus

$$\begin{aligned}
\overline{|\mathscr{M}|^2} &= e^4\left[1 + \frac{1}{E^2}\left(\mathbf{p}^2\cos^2\theta + m_\mu^2\right)\right] \\
&= e^4\left[1 + \cos^2\theta + \frac{m_\mu^2}{E^2}(1 - \cos^2\theta)\right].
\end{aligned} \tag{9.61}$$

Using the general scattering formula of Eq. (7.106), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+\cos^2\theta+\frac{4m_\mu^2}{s}(1-\cos^2\theta)\right], \quad (9.62)$$

which leads to a total cross-section of

$$\sigma = \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+\frac{2m_\mu^2}{s}\right]. \quad (9.63)$$

Earlier we calculated the cross section for elastic e^-e^+ scattering. If the initial energy is sufficiently high, $s > 4m_\mu^2$, that cross section will be augmented by this inelastic contribution. Thus the total cross section will increase. Creation of heavier particles increase the cross section at even higher energies. This provides a tool for searching for new particles.

9.7 Consequence of gauge invariance

So far, we have performed calculations involving only fermions in the initial and final states. Now we discuss processes which involve photons either in the initial or in the final states, or both.

For any physical process involving a photon with momentum k and polarization vector $\epsilon^\mu(k)$ in the final state, the Feynman amplitude may be written in the form $\mathscr{M} = M_\mu \epsilon^{*\mu}(k)$. We now refer back to the calculation of the S-matrix elements in Ch. 6 and recall how this amplitude arises. Let us suppose that the photon was created at the space-time point x. Then in the terms in the S-matrix, we must have made use of a term $\int d^4x\, \mathscr{H}_I(x)$, apart from any other terms at any other space-time points that may have entered the calculation. In other words, the S-matrix element must have arisen from

$$\left\langle f';\gamma(k)\left|\left(-i\int d^4x\, j_\mu(x)A^\mu(x)\right)\cdots\right|i\right\rangle, \quad (9.64)$$

where $\gamma(k)$ denotes the photon and f' denotes any other particle that may be present in the final state, whereas the dots denote any other factors of the interaction Hamiltonian that might be present. Once we use the analog of the one-particle states shown in §6.2, we will obtain

$$-i\epsilon^{*\mu}(k)\int d^4x\, \frac{e^{ik\cdot x}}{\sqrt{2\omega V}}\left\langle f'\left|j_\mu(x)\cdots\right|i\right\rangle. \quad (9.65)$$

Thus the quantity we called M_μ must have arisen from the quantity inside the integral sign, after factoring out the initial and final state factors.

Now imagine replacing $\epsilon^{*\mu}(k)$ with k^μ in the expression above. This will give

$$\begin{aligned} -ik^\mu \int d^4x \, \frac{e^{ik\cdot x}}{\sqrt{2\omega V}} \langle f' | j_\mu(x) \cdots | i \rangle \\ = -\int d^4x \, \frac{(\partial^\mu e^{ik\cdot x})}{\sqrt{2\omega V}} \langle f' | j_\mu(x) \cdots | i \rangle \\ = \int d^4x \, \frac{e^{ik\cdot x}}{\sqrt{2\omega V}} \langle f' | \partial^\mu j_\mu(x) \cdots | i \rangle \, . \end{aligned} \tag{9.66}$$

The last step is obtained by partial integration. Since the other factors, represented by the dots, are not at the point x, the derivative does not act on them. This shows that the quantity under consideration vanishes since $\partial^\mu j_\mu(x) = 0$. Thus if we replace the polarization vector in an S-matrix element by the corresponding momentum vector, the amplitude goes to zero. In other words,

$$k^\mu M_\mu = 0 \, . \tag{9.67}$$

Similarly, we can prove that the same relation would hold if there is a photon in the initial state. Said in another way, if we make the substitution

$$\epsilon_\mu(k) \to \epsilon_\mu(k) + k_\mu \vartheta \tag{9.68}$$

in the amplitude, the co-efficient of ϑ should be zero.

For physical photons, the consequence of using the Lorenz gauge condition translates in momentum space to the relation

$$k \cdot \epsilon = 0 \, . \tag{9.69}$$

So the substitution of Eq. (9.68) does not affect the normalization of the polarization vectors for physical photons, given in Eq. (8.51).

9.8 Compton scattering

Compton scattering is the name given to the elastic scattering of electron and photon:

$$e^-(p,s) + \gamma(k,r) \to e^-(p',s') + \gamma(k',r') \, . \tag{9.70}$$

In the parentheses after the electrons, we have put the notations for their momenta and spins. After the photons, we have indicated their momenta and polarizations. The lowest order diagrams for this process are given in Fig. 9.7. There are two diagrams, marked (a) and (b).

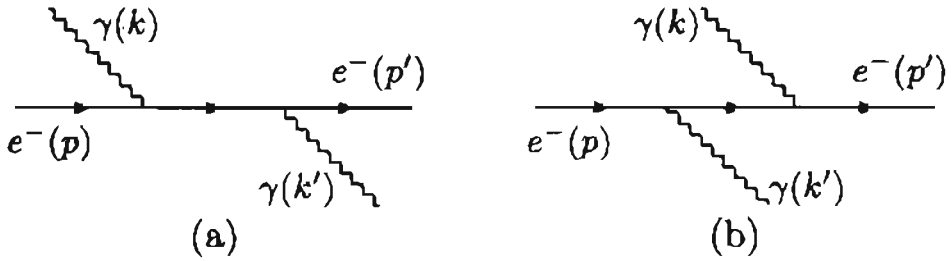


Figure 9.7: Lowest order diagrams for Compton scattering in QED.

The Feynman amplitude

The amplitudes for the two diagrams can be written as

$$i\mathscr{M}_a = \left[\overline{u}(p')(ie\gamma^\mu)\frac{i}{\not p + \not k - m}(ie\gamma^\nu)u(p)\right]\epsilon^*_{r'\mu}(k')\epsilon_{r\nu}(k)\,,$$
$$i\mathscr{M}_b = \left[\overline{u}(p')(ie\gamma^\nu)\frac{i}{\not p - \not k' - m}(ie\gamma^\mu)u(p)\right]\epsilon^*_{r'\mu}(k')\epsilon_{r\nu}(k)\,. \quad (9.71)$$

Summing up the two contributions, we obtain

$$\begin{aligned}\mathscr{M} &= -e^2\overline{u}(p')\left[\not\epsilon'^*\frac{1}{\not p + \not k - m}\not\epsilon + \not\epsilon\frac{1}{\not p - \not k' - m}\not\epsilon'^*\right]u(p)\\ &= -e^2\overline{u}(p')\left[\not\epsilon'^*\frac{\not p + \not k + m}{(p+k)^2 - m^2}\not\epsilon + \not\epsilon\frac{\not p - \not k' + m}{(p-k')^2 - m^2}\not\epsilon'^*\right]u(p)\end{aligned} \quad (9.72)$$

for the total Feynman amplitude, where we have used the shorthands

$$\epsilon_\nu \equiv \epsilon_{r\nu}(k)\,, \qquad \epsilon'_\mu \equiv \epsilon_{r'\mu}(k')\,. \quad (9.73)$$

We note an interesting point here. If we make the substitutions

$$k \to -k'\,, \qquad \epsilon \to \epsilon'^*\,, \quad (9.74)$$

in the amplitude of Eq. (9.72), it is invariant. Physically, this means that if we interchange the photons in the outer lines, the amplitude is unaffected. This is the result of the indistinguishability of photons, and such symmetries are called *crossing symmetries*.

Gauge invariance

The consequence of gauge invariance has been discussed in §9.7. Let us first verify it for the present amplitude. We rewrite Eq. (9.72), shifting the polarization vectors by a term proportional to the corresponding momentum. Denoting the resulting amplitude by $\mathscr{M}'$, we can write

$$\mathscr{M}' = -e^2\overline{u}(p')\Big[(\not\epsilon'^* + \not k'\vartheta'^*)\ \frac{1}{\not p + \not k - m}(\not\epsilon + \not k\vartheta) + (\not\epsilon + \not k\vartheta)\frac{1}{\not p - \not k' - m}(\not\epsilon'^* + \not k'\vartheta'^*)\Big]u(p)\,. \quad (9.75)$$

Thus,

$$\mathscr{M}' - \mathscr{M} = -e^2\overline{u}(p')\left[\vartheta\left(\not\epsilon'^*\frac{1}{\not p + \not k - m}\not k + \not k\frac{1}{\not p - \not k' - m}\not\epsilon'^*\right) + \vartheta'^*\left(\not k'\frac{1}{\not p + \not k - m}\not\epsilon + \not\epsilon\frac{1}{\not p - \not k' - m}\not k'\right) + \vartheta\,\vartheta'^*\left(\not k'\frac{1}{\not p + \not k - m}\not k + \not k\frac{1}{\not p - \not k' - m}\not k'\right)\right]u(p)\,. \quad (9.76)$$

Let us look, for example, at the co-efficient of ϑ. In the first term, we can write

$$\not k = (\not p + \not k - m) - (\not p - m)\,. \quad (9.77)$$

Since this expression operates on $u(p)$ to the right, the second parenthesis gives zero while acting on it. The other part cancels the propagator, and so we are left with just $\not\epsilon'^*$. Similarly, in the second term, we can write

$$\not k = (\not p' - m) - (\not p' - \not k - m)\,. \quad (9.78)$$

Here also, the first parenthesis vanishes because of the spinor on its left. The other part cancels the propagator but for a sign, using $p + k = p' + k'$ from momentum conservation. Thus, this cancels the first term. One can similarly see that the co-efficient of ϑ'^* vanishes, and so does the co-efficient of $\vartheta\vartheta'^*$.

Choice of gauge and frame

Having established gauge invariance, we can use it to our advantage. For example, given a frame, we can choose any one component of the

polarization vectors to be zero by using this freedom. We will choose

$$\epsilon^\mu = (0, \boldsymbol{\epsilon}), \qquad \epsilon'^\mu = (0, \boldsymbol{\epsilon}') \tag{9.79}$$

in the Lab frame in which the initial electron is at rest. The advantage of this choice is that

$$p \cdot \epsilon = 0 = p \cdot \epsilon', \tag{9.80}$$

which make the expressions particularly simple.

For example, the first term in Eq. (9.72) contains the expression

$$\begin{aligned}(\not{p} + m)\not{\epsilon} u(p) &= (p^\mu \epsilon^\nu \gamma_\mu \gamma_\nu + m\epsilon^\nu \gamma_\nu) u(p) \\ &= [p^\mu \epsilon^\nu (2g_{\mu\nu} - \gamma_\nu \gamma_\mu) + m\not{\epsilon}]\, u(p) \\ &= [2p \cdot \epsilon + \not{\epsilon}(-\not{p} + m)]\, u(p) \\ &= 2p \cdot \epsilon\, u(p)\,. \end{aligned} \tag{9.81}$$

This derivation makes no assumption about the choice of the polarization vector or frame. If however we stick to the choice of Eq. (9.80), this expression vanishes. The argument is no different if the polarization vector on the right is ϵ'. So the amplitude of Eq. (9.72) can be written as

$$\mathscr{M} = -e^2 \bar{u}(p') F u(p)\,, \tag{9.82}$$

where

$$F = \frac{\not{\epsilon}' \not{k} \not{\epsilon}}{2p \cdot k} + \frac{\not{\epsilon} \not{k}' \not{\epsilon}'}{2p \cdot k'}\,. \tag{9.83}$$

We have used the on-shell conditions

$$k^2 = k'^2 = 0\,, \qquad p^2 = p'^2 = m^2\,, \tag{9.84}$$

to simplify the denominators, and assumed the polarization vectors to have real components for the sake of simplicity.

We will derive the cross-section in the Lab frame for specific polarizations for the initial and the final photons. The averaging over the initial electron spin will be done by summing over the spins and dividing by 2. Thus,

$$\begin{aligned}\overline{|\mathscr{M}|^2} &= \frac{1}{2} \sum_{\text{spin}} |\mathscr{M}|^2 \\ &= \frac{1}{2} e^4 \,\text{Tr}\left[(\not{p}' + m) F (\not{p} + m) F^\dagger\right]\,, \end{aligned} \tag{9.85}$$

and in this case,

$$F^{\updownarrow} = \frac{\not\epsilon \not k \not\epsilon'}{2p\cdot k} + \frac{\not\epsilon' \not k' \not\epsilon}{2p\cdot k'} \,. \tag{9.86}$$

Since in the Lab frame

$$p\cdot k = m\omega\,, \qquad p\cdot k' = m\omega'\,, \tag{9.87}$$

where ω and ω' are the initial and final photon energies, we can write

$$\overline{|\mathscr{M}|^2} = \frac{e^4}{8m^2}\left[\frac{T_{11}}{\omega^2} + \frac{T_{22}}{\omega'^2} + \frac{T_{12}+T_{21}}{\omega\omega'}\right] . \tag{9.88}$$

In this formula, T_{11} through T_{21} are various traces, which we evaluate now.

Calculation of the traces

We start with T_{11}, which is given by

$$\begin{aligned} T_{11} &= \mathrm{Tr}\,[(\not p' + m)\not\epsilon' \not k \not\epsilon (\not p + m)\not\epsilon \not k \not\epsilon'] \\ &= \mathrm{Tr}\,[\not p' \not\epsilon' \not k \not\epsilon \not p \not\epsilon \not k \not\epsilon'] + m^2 \mathrm{Tr}\,[\not\epsilon' \not k \not\epsilon \not\epsilon \not k \not\epsilon'] \,. \end{aligned} \tag{9.89}$$

We first show that the term proportional to m^2 is zero. For physical states of polarization, we can use $\not\epsilon\not\epsilon = \epsilon^2 = -1$, as given in Eq. (8.51). This enables us to write

$$\begin{aligned} \mathrm{Tr}\,[\not\epsilon' \not k \not\epsilon \not\epsilon \not k \not\epsilon'] &= -\,\mathrm{Tr}\,[\not\epsilon' \not k \not k \not\epsilon'] \\ &= -k^2\,\mathrm{Tr}\,[\not\epsilon' \not\epsilon'] = 0\,, \end{aligned} \tag{9.90}$$

using Eq. (9.84).

As for the other trace in Eq. (9.89), we can use the relation of Eq. (4.65) repeatedly to reduce it, and then use Eqs. (9.80) and (9.84). This gives

$$\begin{aligned} T_{11} &= 2p\cdot\epsilon\,\mathrm{Tr}\,[\not p' \not\epsilon' \not k \not\epsilon \not k \not\epsilon'] - \mathrm{Tr}\,[\not p' \not\epsilon' \not k \not p \not\epsilon \not k \not\epsilon'] \\ &= \mathrm{Tr}\,[\not p' \not\epsilon' \not k \not p \not k \not\epsilon'] \\ &= 2p\cdot k\,\mathrm{Tr}\,[\not p' \not\epsilon' \not k \not\epsilon'] - \mathrm{Tr}\,[\not p' \not\epsilon' \not p \not k \not k \not\epsilon'] \,. \end{aligned} \tag{9.91}$$

Again, $\not k \not k = k^2 = 0$, so the second trace vanishes, and we are left with only the first term, which gives

$$T_{11} = 8p\cdot k\,\left[2p'\cdot\epsilon' k\cdot\epsilon' + p'\cdot k\right] . \tag{9.92}$$

Using Eqs. (9.69) and (9.80), we can write $p'\cdot\epsilon' = (p+k-k')\cdot\epsilon' = k\cdot\epsilon'$. Also, using momentum conservation, we find $p'\cdot k = p\cdot k'$. Thus, finally we can write

$$T_{11} = 8m\omega \left[2(k\cdot\epsilon')^2 + m\omega'\right] , \tag{9.93}$$

using Eq. (9.87).

It is now easy to evaluate T_{22}, which is given by

$$T_{22} = \mathrm{Tr}\ [(\not{p}' + m)\not{\epsilon}\not{k}'\not{\epsilon}'(\not{p} + m)\not{\epsilon}'\not{k}'\not{\epsilon}] . \tag{9.94}$$

This is related to T_{11} by crossing symmetry mentioned in Eq. (9.74). Thus we can immediately write down the result of the trace, using crossing symmetry on the final result for T_{11}. This gives

$$T_{22} = -8m\omega' \left[2(k'\cdot\epsilon)^2 - m\omega\right] . \tag{9.95}$$

We are now left with T_{12} and T_{21}, which are defined as

$$\begin{aligned} T_{12} &= \mathrm{Tr}\ [(\not{p}' + m)\not{\epsilon}'\not{k}\not{\epsilon}(\not{p} + m)\not{\epsilon}'\not{k}'\not{\epsilon}] , \\ T_{21} &= \mathrm{Tr}\ [(\not{p}' + m)\not{\epsilon}\not{k}'\not{\epsilon}'(\not{p} + m)\not{\epsilon}\not{k}\not{\epsilon}'] . \end{aligned} \tag{9.96}$$

A useful result, shown in Eq. (A.28) of Appendix A, is that the trace of a string of γ-matrices is unaffected if we take the string in reverse order. Thus

$$T_{21} = \mathrm{Tr}\ [\not{\epsilon}'\not{k}\not{\epsilon}(\not{p} + m)\not{\epsilon}'\not{k}'\not{\epsilon}(\not{p}' + m)] , \tag{9.97}$$

which is the same as T_{12} because of cyclicity of trace. So we can evaluate either one of them. For example, starting with the expression for T_{12} and using momentum conservation, $p' = p + k - k'$, we can write

$$\begin{aligned} T_{12} = {} & \mathrm{Tr}\ [(\not{p} + m)\not{\epsilon}'\not{k}\not{\epsilon}(\not{p} + m)\not{\epsilon}'\not{k}'\not{\epsilon}] \\ & + \mathrm{Tr}\ [\not{k}\not{\epsilon}'\not{k}\not{\epsilon}(\not{p} + m)\not{\epsilon}'\not{k}'\not{\epsilon}] - \mathrm{Tr}\ [\not{k}'\not{\epsilon}'\not{k}\not{\epsilon}(\not{p} + m)\not{\epsilon}'\not{k}'\not{\epsilon}]. \end{aligned} \tag{9.98}$$

In the last two traces, we can drop the mass term in the parentheses since it involves an odd number of γ-matrices. Then using the cyclic property of trace and the relations in Eqs. (9.69), (9.80) and (9.84),

we can reduce the second trace to the following form:

$$\begin{aligned}
\text{2nd term in } T_{12} &= \text{Tr}\left[(2k\cdot\epsilon' - \not\epsilon'\not k)\not k\not\epsilon\not p\not\epsilon'\not k'\not\epsilon\right] \\
&= 2k\cdot\epsilon'\ \text{Tr}\left[\not k\not\epsilon\not p\not\epsilon'\not k'\not\epsilon\right] \\
&= -2k\cdot\epsilon'\ \text{Tr}\left[\not\epsilon\not k\not p\not\epsilon'\not k'\not\epsilon\right] \\
&= -2k\cdot\epsilon'\ \text{Tr}\left[\not k\not p\not\epsilon'\not k'\not\epsilon\not\epsilon\right] \\
&= 2k\cdot\epsilon'\ \text{Tr}\left[\not k\not p\not\epsilon'\not k'\right] \\
&= -8(k\cdot\epsilon')^2\ p\cdot k' \\
&= -8m\omega'(k\cdot\epsilon')^2\,. \qquad (9.99)
\end{aligned}$$

Following exactly similar steps, we can find that

$$\text{3rd term in } T_{12} = -8m\omega(k'\cdot\epsilon)^2\,. \qquad (9.100)$$

As for the first term in T_{12}, we can reduce it by writing

$$\begin{aligned}
(\not p + m)\not\epsilon'\not k\not\epsilon(\not p + m) &= \not\epsilon'(-\not p + m)\not k(-\not p + m)\not\epsilon \\
&= \not\epsilon'\left[-2p\cdot k + \not k(\not p + m)\right](-\not p + m)\not\epsilon\,, \\
&= 2p\cdot k\ \not\epsilon'(\not p - m)\not\epsilon\,, \qquad (9.101)
\end{aligned}$$

where we have used Eqs. (9.80) and (4.65). Putting this back and noting that the term proportional to m has an odd number of γ-matrices, we obtain

$$\begin{aligned}
\text{1st term in } T_{12} &= 2p\cdot k\ \text{Tr}\left[\not\epsilon'\not p\not\epsilon\not\epsilon'\not k'\not\epsilon\right] \\
&= 2p\cdot k\ \text{Tr}\left[\not\epsilon'\not\epsilon\not p\not k'\not\epsilon'\not\epsilon\right] \\
&= 2p\cdot k\ \text{Tr}\left[(2\epsilon\cdot\epsilon' - \not\epsilon\not\epsilon')\not p\not k'\not\epsilon'\not\epsilon\right] \\
&= 4p\cdot k\ \epsilon\cdot\epsilon'\ \text{Tr}\left[\not p\not k'\not\epsilon'\not\epsilon\right] - 2p\cdot k\ \text{Tr}\left[\not p\not k'\right] \\
&= 16p\cdot k\ p\cdot k'\ (\epsilon\cdot\epsilon')^2 - 8p\cdot k\ p\cdot k' \\
&= 8m^2\omega\omega'\Big[2(\epsilon\cdot\epsilon')^2 - 1\Big]\,. \qquad (9.102)
\end{aligned}$$

Adding all the contributions, we obtain

$$T_{12} = T_{21} = 8m^2\omega\omega'\left[2(\epsilon\cdot\epsilon')^2 - 1\right] - 8m\omega'(k\cdot\epsilon')^2 + 8m\omega(k'\cdot\epsilon)^2\,. \qquad (9.103)$$

Putting the results for T_{11} through T_{21} into Eq. (9.88), we get

$$\overline{|\mathcal{M}|^2} = e^4\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\epsilon\cdot\epsilon')^2 - 2\right]\,. \qquad (9.104)$$

The differential cross-section

Substituting this in the general formula of Eq. (7.118) for 2-to-2 scattering obtained in the lab frame, we obtain

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2(m+\omega-\omega\cos\theta)^2}\,\overline{|\mathscr{M}|^2}\,, \tag{9.105}$$

where θ is the angle at which the photon is scattered, i.e., the angle between the directions of the incoming and the outgoing photons.

Noting that the energy of the scattered photon is given by

$$\omega' = \frac{\omega}{1+(\omega/m)(1-\cos\theta)}\,, \tag{9.106}$$

which follows from the general expression given in Eq. (7.115), we can write the differential cross-section as

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+4(\epsilon\cdot\epsilon')^2-2\right]. \tag{9.107}$$

This is called the Klein-Nishina formula, after the people who derived it.

The total cross-section can be obtained by integrating over the angular variables after we substitute ω' from Eq. (9.106). But there is a word of caution. In the expression for the differential cross-section, the dot product of the initial and final polarization vectors appear. This may also depend on the angle θ. Below, we calculate this dependence explicitly in a specific case.

Differential cross-section for unpolarized photons

If our initial photon is unpolarized and we do not measure the polarization of the final photon, we should average over initial polarizations and sum over final ones. Since there are two physical states of polarization, this means that we should sum over initial and final polarizations, and divide the result by 2. For all terms which are independent of the polarization vectors, this procedure will give an overall factor of 2. For the polarization dependent expression $(\epsilon\cdot\epsilon')^2$, it is better to revert back to the longer notation which was given up in Eq. (9.73). We need to evaluate

$$\frac{1}{2}\sum_{r,r'}\left[\epsilon_r(k)\cdot\epsilon_{r'}(k')\right]^2 = \frac{1}{2}\Big[\sum_r \epsilon_{ri}(k)\epsilon_{rj}(k)\Big]\,\Big[\sum_{r'}\epsilon_{r'i}(k')\epsilon_{r'j}(k')\Big]\,. \tag{9.108}$$

For either of these sums, we can take a look at the various choices of polarization vectors given in §8.5, which give

$$\sum_r \epsilon_{ri}(k)\epsilon^*_{rj}(k) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\,, \tag{9.109}$$

where the sum is over transverse, i.e., the physical polarization states only. Using this for the real polarization vectors that we are considering here, we obtain finally

$$\frac{1}{2}\sum_{r,r'}\left[\epsilon_r(k)\cdot\epsilon_{r'}(k')\right]^2 = \frac{1}{2}(1+\cos^2\theta)\,, \tag{9.110}$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. So we obtain

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{unpol}} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right]\,. \tag{9.111}$$

□ **Exercise 9.7** Derive the differential cross section for unpolarized photons using the polarization sum given in Eq. (8.88). [Note: You cannot use Eq. (9.79) at the same time.]

Total cross-section for unpolarized photons

The integration over angular variables is now straightforward, and it gives

$$\sigma_{\text{unpol}} = \frac{2\pi\alpha^2}{m^2}\left[\frac{1+r}{(1+2r)^2}+\frac{2}{r^2}-\frac{2(1+r)-r^2}{2r^3}\ln(1+2r)\right]. \tag{9.112}$$

where $r = \omega/m$. If $r \ll 1$, i.e., if the incident photon has very low energy, then this reduces to

$$\sigma_{\text{unpol}} = \frac{8\pi\alpha^2}{3m^2} = 0.665\times10^{-24}\,\text{cm}^2\,. \tag{9.113}$$

The cross-section in this limit is called the Thomson cross-section. This is often taken as a benchmark value for electromagnetic cross-sections. Eq. (9.113) can be deduced using the methods of classical electromagnetic theory, assuming that the frequency of the incident radiation does not change in the process of scattering. As seen from Eq. (9.106), this is a good approximation if $\omega \ll m$.

9.9 Scattering by an external field

So far we have talked about the electromagnetic field as a dynamical entity which can be quantized in terms of photon creation and annihilation operators. What happens if we have an electron moving through a field such that the back reaction of the electron on the field can be neglected? In such cases the 'external' field can be described safely by classical functions.

Suppose we have a heavy nucleus which scatters an incoming electron beam. We can treat the electromagnetic field of the nucleus as a static Coulomb field in the rest frame of the nucleus. If the nucleus is sufficiently heavy, we can ignore the momentum imparted to it by the scattered electron. Let us then split the vector potential into an 'external' part $A_{e\mu}$ and a dynamical part A_μ, and write the interaction Lagrangian as

$$e :\overline{\psi}(\not{A} + \not{A}_e)\psi: \,. \tag{9.114}$$

Nothing new has been done yet.

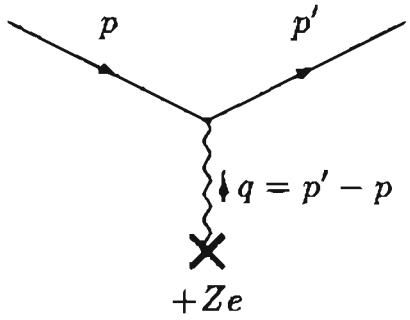


Figure 9.8: Scattering of electron by a nucleus of charge $+Ze$.

However, neglecting the change of momentum of the nucleus means that we can get a nonzero first order process out of this. Let us be specific and consider the situation where an electron of 4-momentum p is scattered by a static Coulomb field. The first order term in the S-matrix in this case is

$$S^{(1)} = ie \int d^4x \, \overline{\psi}_-(\not{A} + \not{A}_e)\psi_+ \,. \tag{9.115}$$

We know from §9.3 that the dynamical part does not contribute because of energy-momentum conservation. So we are left with only

the contribution from the static 'external' part,

$$S_e^{(1)} = ie \int d^4x \, \bar{\psi}_- \not{A}_e \psi_+ \,. \tag{9.116}$$

Since the external field is independent of time, we can write it as

$$A_e^\mu \equiv A_e^\mu(\boldsymbol{x}) = \int \frac{d^3q}{(2\pi)^3} \, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} A_e^\mu(\boldsymbol{q}) \,. \tag{9.117}$$

So the transition amplitude is

$$\begin{aligned}
\left\langle e^-(p') \left| S_e^{(1)} \right| e^-(p) \right\rangle &= ie \int d^4x \frac{1}{\sqrt{2E'V}} \bar{u}(p') e^{ip'\cdot x} \\
&\qquad \times \left(\int \frac{d^3q}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \not{A}_e(q) \right) \frac{1}{\sqrt{2EV}} e^{-ip\cdot x} u(p) \\
&= ie \frac{2\pi\delta(E-E')}{\sqrt{2EV}\sqrt{2E'V}} \int d^3q \; \delta^3(\boldsymbol{q} - \boldsymbol{p}' + \boldsymbol{p}) \bar{u}(p') \not{A}_e(q) u(p) \\
&= \frac{2\pi\delta(E-E')}{\sqrt{2EV}\sqrt{2E'V}} i\mathscr{M}_{fi} \,,
\end{aligned} \tag{9.118}$$

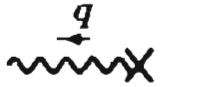
where

$$\mathscr{M}_{fi} = e\bar{u}(p') \not{A}_e(q = p' - p) u(p) \,. \tag{9.119}$$

The S-matrix element can be viewed as in Ch. 6 provided we make the following adjustments:

1. In the overall kinematic factors appearing in the S-matrix, replace $(2\pi)^4\delta^4(p_i - p_f)$ by $(2\pi)\delta(E_i - E_f)$. This implies that the 4-momentum is not conserved at the vertex. This is not a failure of the theory. Rather, it means that we are ignoring the 3-momentum transferred to the nucleus.

2. In the Feynman amplitude part, for each interaction vertex with an external static field write a factor



$$A_{e\mu}(q) \,. \tag{9.120}$$

□ **Exercise 9.8** The nucleus is at rest in the Lab frame. If the energy E of the initial electron is negligible compared to the mass M of the nucleus, show from the kinematics that the energy E' of the final electron is approximately equal to the initial energy.

Putting this back into Eq. (9.118) and squaring the S-matrix element, we obtain the transition rate to be

$$|S_{fi}|^2/T = \frac{\delta(E-E')}{2E'V\,2EV}\overline{|\mathscr{M}|^2}, \tag{9.121}$$

where we have used the arguments leading to Eq. (6.17) to put

$$|\delta(E-E')|^2 = \frac{T}{2\pi}\delta(E-E'), \tag{9.122}$$

T being the time in which we perform all the calculations. The cross section can then be written, in analogy with Eq. (7.88), as

$$d\sigma = \frac{1}{2\mathbf{p}}\frac{d^3p'}{(2\pi)^3 2E'}(2\pi)\delta(E-E')\overline{|\mathscr{M}|^2}, \tag{9.123}$$

where the factor of $2\mathbf{p}$ sitting in front appears as a product of $2E$ which comes from the initial state factor, and the velocity of the incoming particle, which is $\mathbf{p}/E$. The energy conserving δ-function ensures $\mathbf{p}=\mathbf{p}'$. Integration over the magnitude of $\boldsymbol{p}'$ gives

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}\overline{|\mathscr{M}|^2}. \tag{9.124}$$

Let us calculate the cross-section of scattering of an electron from a nucleus of charge Ze using these rules. The external field corresponding to the nucleus is the four-vector

$$A_e^\mu(x) = \left(\frac{Ze}{4\pi r}, 0, 0, 0\right). \tag{9.125}$$

The Fourier transform of this is

$$A_e^\mu(q) = \left(\frac{Ze}{\mathbf{q}^2}, 0, 0, 0\right). \tag{9.126}$$

Then,

$$\begin{aligned}\overline{|\mathscr{M}|^2} &= \frac{1}{2}\left(\frac{Ze^2}{\mathbf{q}^2}\right)^2 \sum_{r,s}\left|\overline{u}(p')\gamma_0 u(p)\right|^2 \\ &= \frac{8\pi^2(Z\alpha)^2}{\mathbf{q}^4}\,\mathrm{Tr}\left[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\right], \\ &= \frac{32\pi^2(Z\alpha)^2}{\mathbf{q}^4}\left[EE'+\mathbf{p}\cdot\mathbf{p}'+m^2\right]. \end{aligned} \tag{9.127}$$

Denoting the angle between p' and p by θ, we have

$$p \cdot p' = \mathbf{p}^2 \cos\theta, \qquad \mathbf{q}^2 = |p' - p|^2 = 4\mathbf{p}^2 \sin^2\frac{\theta}{2}. \tag{9.128}$$

Therefore,

$$\begin{aligned}\overline{|\mathscr{M}|^2} &= \frac{2\pi^2(Z\alpha)^2}{\mathbf{p}^4 \sin^4\frac{\theta}{2}}\left[E^2 + \mathbf{p}^2\cos\theta + m^2\right] \\ &= \frac{4\pi^2(Z\alpha)^2}{\mathbf{p}^4 \sin^4\frac{\theta}{2}}\left[E^2 - \mathbf{p}^2\sin^2\frac{\theta}{2}\right].\end{aligned} \tag{9.129}$$

Putting this back in Eq. (9.124), we obtain

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{4\mathbf{p}^4 \sin^4\frac{\theta}{2}}\left[E^2 - \mathbf{p}^2\sin^2\frac{\theta}{2}\right]. \tag{9.130}$$

Since the velocity v is given by $v = \mathbf{p}/E$, we can rewrite the differential cross section as

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{4E^2v^4 \sin^4\frac{\theta}{2}}\left(1 - v^2\sin^2\frac{\theta}{2}\right). \tag{9.131}$$

In the non-relativistic limit, this reduces to the *Rutherford scattering formula*:

$$\frac{d\sigma}{d\Omega} \simeq \frac{(Z\alpha)^2}{4m^2v^4 \sin^4\frac{\theta}{2}}. \tag{9.132}$$

9.10 Bremsstrahlung

Bremsstrahlung, or braking radiation, is the radiation produced by the deceleration of a charge by an electric field, such as that of a heavy nucleus. There are two diagrams at the lowest order of QED which contribute to Bremsstrahlung of an electron in the field of a static nucleus, shown in Fig. 9.9. Both the quantized field and the external field now play a role. However, there is a serious problem with the scattering amplitude as calculated for the process. We run

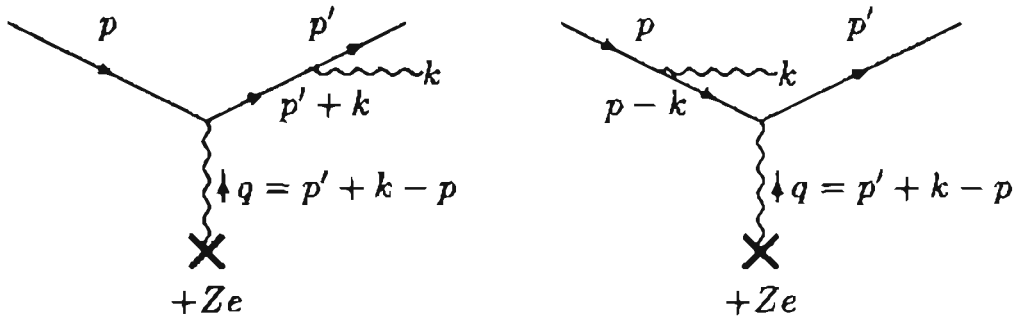


Figure 9.9: The two lowest order diagrams contributing to Bremsstrahlung.

into *infra-red divergence* – the cross-section becomes infinite as the energy of the emitted photon goes to zero. It is as if an electron passing by a heavy nucleus produces a very large number of very low-frequency photons.

The Feynman amplitude for the emission of a photon is

$$i\mathscr{M} = -ie^2\overline{u}(p')\left[\not{\epsilon}(k)\frac{\not{p}' + \not{k} + m}{2p'\cdot k}\not{A}_e(q) + \not{A}_e(q)\frac{\not{p} - \not{k} + m}{-2p\cdot k}\not{\epsilon}(k)\right]u(p)\,. \tag{9.133}$$

The matrix element for the transition is then

$$\langle f\,|S|\,i\rangle = 2\pi\delta(E - E' - \omega)\frac{1}{\sqrt{2EV}}\frac{1}{\sqrt{2E'V}}\frac{1}{\sqrt{2\omega V}}\mathscr{M}\,. \tag{9.134}$$

The final density of states is now $\dfrac{Vd^3p'Vd^3k}{(2\pi)^6}$, and the incident flux is $\dfrac{\mathbf{p}}{EV}$, so that the differential cross-section is

$$d\sigma = \frac{1}{(2\pi)^5 8\omega}\frac{\mathbf{p}'}{\mathbf{p}}\,\overline{|\mathscr{M}|^2}\,d^3k\,d\Omega'\,. \tag{9.135}$$

In the soft photon limit $\omega \approx 0$, using Eq. (9.81) and similar equations, we can write the Feynman amplitude as

$$\begin{aligned} i\mathscr{M} &\approx -ie^2\overline{u}(p')\not{A}_e(q)u(p)\left[\frac{p'\cdot\epsilon}{p'\cdot k} - \frac{p\cdot\epsilon}{p\cdot k}\right] \\ &= -ie\mathscr{M}_0\left[\frac{p'\cdot\epsilon}{p'\cdot k} - \frac{p\cdot\epsilon}{p\cdot k}\right], \end{aligned} \tag{9.136}$$

where $\mathscr{M}_0$ is the Feynman amplitude for scattering without emission, Eq. (9.119). This gives

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{\alpha}{(2\pi)^2} \int_{\rm IR} \frac{d^3k}{\omega} \left[\frac{p' \cdot \epsilon}{p' \cdot k} - \frac{p \cdot \epsilon}{p \cdot k}\right]^2 + \cdots, \quad (9.137)$$

where the integration is limited to small values of ω, since this is the region we are interested in. This is indicated by the subscript 'IR' on the integral, and the dots stand for the contribution from larger ω. The differential cross-section (as well as the Feynman amplitude) clearly diverges as $\omega \to 0$. This is called the *infra-red divergence*.

□ **Exercise 9.9** Show that the cross-section, Eq. (9.137), when summed over all polarizations, is

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{-\alpha}{(2\pi)^2}\right) \int_{\rm IR} \frac{d^3k}{\omega} \left(\frac{p'}{p' \cdot k} - \frac{p}{p \cdot k}\right)^2 . \quad (9.138)$$

One way this divergence can be understood is by noting that when $k \approx 0$, the momentum of the internal electron is $k + p' \approx p'$. But since $p'^2 = m^2$, the internal propagator is divergent. Physically, what we find here is that it is possible to fit in an infinite number of zero energy photons in the radiation.

This is not a problem specific to this process. In fact, Bremsstrahlung can take place from a charged external line in any process. For example, in electron-electron scattering, a photon can be emitted from any of the external lines. Such processes also suffer from the same infra-red divergence problem. The infra-red problem has its roots in the classical theory, but it is quantum field theory that comes to the rescue. Although the divergence shows up at the lowest order quantum calculation, the removal of the divergence requires higher order corrections. This will be discussed in Ch. 12.

Chapter 10

P, T, C and their combinations

As we mentioned before, the symmetries of a physical system provide important clues to the structure of the Lagrangian. So far, whenever we talked about symmetries, we meant continuous symmetries. In Ch. 2, for example, we discussed the consequences of invariance of a Lagrangian under such continuous symmetries and derived Noether's theorem. We applied Noether's theorem in various circumstances in subsequent chapters. Finally, in Ch. 9, we discussed how a continuous symmetry can be realized as a local symmetry if some gauge fields, like the electromagnetic field, is present.

There is another class of symmetries, called *discrete symmetries*. The name implies that the transformation relating to these symmetries cannot be viewed as the continuous change of a variable. Rather, a discrete operation is involved. Some such symmetries are extremely important in understanding the physical content of a theory and restricting the form of interactions in the theory. In this chapter, we discuss some symmetries of this sort and discuss their physical implications.

10.1 Motivations from classical physics

We begin our discussion by noting how some discrete symmetries are inherent in the laws of classical physics. For this, consider the equation of motion of a non-relativistic test particle of mass m and charge q in the Coulomb field of a particle of charge q':

$$m\frac{d^2\boldsymbol{x}}{dt^2} = \frac{qq'}{4\pi r^3}\boldsymbol{x}\,, \tag{10.1}$$

where $r = |\boldsymbol{x}|$, and we took the origin of our co-ordinate system at the position of the charge q'. Note that if we make the substitution

$$\boldsymbol{x} \rightarrow -\boldsymbol{x}\,, \tag{10.2}$$

this equation remains invariant. This operation is called *parity* operation, and therefore parity is a symmetry of Eq. (10.1). Another discrete symmetry which keeps the same equation invariant is *time reversal*, which is defined to be the transformation

$$t \rightarrow -t\,. \tag{10.3}$$

Since Eq. (10.1) involves a double derivative of time on the left hand side, it is unaffected by the change of sign of time.

Apart from having parity and time reversal symmetries, Eq. (10.1) is also invariant if we change the sign of the charges of each particle:

$$q \rightarrow -q\,, \qquad q' \rightarrow -q'\,. \tag{10.4}$$

This symmetry is called *charge conjugation* symmetry. In the rest of this chapter, we will discuss how these symmetries can be implemented into quantum field theory.

□ **Exercise 10.1** Verify that parity and time reversal are Lorentz transformations.

10.2 Parity

10.2.1 Free scalar fields

Let us start with free Lagrangians and see whether they respect the parity symmetry. Consider first the free Lagrangian of a real scalar field, Eq. (3.5). We write it more explicitly here, showing the time and the space derivatives separately:

$$\mathscr{L}(x) = \frac{1}{2}\left[(\partial_t\phi(x))^2 - (\boldsymbol{\nabla}_{\boldsymbol{x}}\phi(x))^2 - m^2\phi^2(x)\right]\,. \tag{10.5}$$

The parity transformation is just measuring things using a system of co-ordinates

$$\tilde{x} = (t, -\boldsymbol{x})\,. \tag{10.6}$$

Since $\tilde{x}$ is related to co-ordinates x by a special kind of Lorentz transformation, let us try to represent the parity transformation also as a constant linear transformation on the fields,

$$\phi_P(x) \equiv \mathcal{P}\phi(x)\mathcal{P}^{-1} = \eta_P \phi(\tilde{x}) \,, \tag{10.7}$$

where $\mathcal{P}$ is the parity operator and η_P is independent of space-time. If we write the Lagrangian in terms of this parity-transformed field, but in the original co-ordinates x, the result should be the same as writing the Lagrangian in terms of the original fields, but co-ordinates $\tilde{x}$. In other words, parity invariance means

$$\mathcal{P}\mathscr{L}(x)\mathcal{P}^{-1} = \mathscr{L}(\tilde{x}) \,. \tag{10.8}$$

So the action remains invariant. Now,

$$\mathcal{P}\mathscr{L}(x)\mathcal{P}^{-1} = \frac{1}{2}\left[(\partial_t \phi_P(x))^2 - (\boldsymbol{\nabla}_{\boldsymbol{x}}\phi_P(x))^2 - m^2\phi_P^2(x)\right] , \tag{10.9}$$

and Eq. (10.8) holds provided $\eta_P = \pm 1$, i.e.,

$$\phi_P(x) \equiv \mathcal{P}\phi(x)\mathcal{P}^{-1} = \pm\phi(\tilde{x}) \,. \tag{10.10}$$

The factor η_P is called the *intrinsic parity* of the particle corresponding to the field ϕ, whose values can be ± 1. For a complex scalar field, we obtain in the same way that η_P can be any arbitrary phase. But parity operation done twice is the identity operation, which restricts η_P to the values ± 1. From the free Lagrangian, there is no way to choose between these two possibilities for intrinsic parity. We will see later that this need not be the case when interactions are introduced.

□ **Exercise 10.2** Calculate the effect of the parity operator on the creation and annihilation operators of ϕ. Assuming that the vacuum is parity invariant, show that the one-particle states transform as

$$\mathcal{P}\,|\boldsymbol{k}\rangle = \eta_P\,|-\boldsymbol{k}\rangle \,. \tag{10.11}$$

10.2.2 Free Dirac field

For fermions the situation is a bit more involved because a Dirac fermion has four components. We start from the Dirac Lagrangian of Eq. (4.84), written in a more explicit notation:

$$\mathscr{L}(x) = \overline{\psi}(x)\left(i\gamma_0\partial_t + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} - m\right)\psi(x) \,. \tag{10.12}$$

As in the case of the scalar field, let us write $\mathcal{P}\psi(x)\mathcal{P}^{-1}$ as a constant linear transform of $\psi(\tilde{x})$. Since parity is a special Lorentz transformation, it will mix the components of ψ like the usual Lorentz transformations,

$$\psi_P(x) \equiv \mathcal{P}\psi(x)\mathcal{P}^{-1} = \mathsf{P}\psi(\tilde{x})\,, \tag{10.13}$$

where P is a 4×4 matrix, independent of space-time.

Our task is now to find out whether there exists a matrix P which will make the Lagrangian parity-invariant, i.e., will ensure Eq. (10.8) for the Dirac Lagrangian. Since $\overline{\psi}_P(x) = \psi_P^\dagger(x)\gamma_0 = \psi^\dagger(\tilde{x})\mathsf{P}^\dagger\gamma_0 = \overline{\psi}(\tilde{x})\gamma_0\mathsf{P}^\dagger\gamma_0$, we get

$$\begin{aligned}\mathcal{P}\mathscr{L}(x)\mathcal{P}^{-1} &= \overline{\psi}(\tilde{x})\gamma_0\mathsf{P}^\dagger\gamma_0\,(i\gamma_0\partial_t + i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} - m)\,\mathsf{P}\psi(\tilde{x}) \\ &= \overline{\psi}(\tilde{x})\gamma_0\mathsf{P}^\dagger\gamma_0\,(i\gamma_0\partial_t - i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}_{-\boldsymbol{x}} - m)\,\mathsf{P}\psi(\tilde{x})\,. \end{aligned}\tag{10.14}$$

This will be the same as the expression for $\mathscr{L}(\tilde{x})$ provided the matrix P satisfies the following conditions:

$$\begin{aligned}\mathsf{P}^\dagger\mathsf{P} &= 1\,, \\ \gamma_0\mathsf{P}^\dagger\gamma_0\gamma_i\mathsf{P} &= -\gamma_i\,, \\ \gamma_0\mathsf{P}^\dagger\gamma_0\mathsf{P} &= 1\,. \end{aligned}\tag{10.15}$$

Of course there is a matrix for which these conditions are satisfied. In fact, if we put $\mathsf{P} = \gamma_0$, all these equations will be satisfied. A more general solution can be written as

$$\mathsf{P} = \eta_P\gamma_0\,, \tag{10.16}$$

where η_P is called the *intrinsic parity* of the fermion. The parity transformation properties of free fermion fields are then

$$\psi_P(x) \equiv \mathcal{P}\psi(x)\mathcal{P}^{-1} = \eta_P\gamma_0\psi(\tilde{x})\,. \tag{10.17}$$

Applying parity transformation again, we find $\eta_P = \pm 1$.

□ **Exercise 10.3** Use the Dirac matrices and the plane-wave solution in the Dirac-Pauli representation to verify that

$$\gamma_0 u_s(\boldsymbol{p}) = u_s(-\boldsymbol{p})\,, \qquad \gamma_0 v_s(\boldsymbol{p}) = -v_s(-\boldsymbol{p})\,. \tag{10.18}$$

Note that these relations must be obeyed in any representation of the Dirac matrices.

□ **Exercise 10.4** Apply the parity transformation rule, Eq. (10.17), on the Fourier expansion of a free fermion field given in Eq. (4.96). Using Eq. (10.18), show that the single particle and antiparticle states transform oppositely under parity.

10.2.3 Free photon field

We now turn to the Lagrangian of the free photon field and examine whether it is invariant under the parity symmetry. For this purpose, let us rewrite the Lagrangian,

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu)\left[(\partial^\mu A^\nu) - (\partial^\nu A^\mu)\right] \\ &= -\frac{1}{2}\Big[(\partial_0 A_i)(\partial^0 A^i - \partial^i A^0) + (\partial_i A_0)(\partial^i A^0 - \partial^0 A^i) \\ &\qquad +(\partial_i A_j)(\partial^i A^j - \partial^j A^i)\Big]\,. \end{aligned} \tag{10.19}$$

In the parity transformed Lagrangian, A^0 and A^i should be replaced by A^0_P and A^i_P respectively. The result shows that the Lagrangian is parity invariant if we define the parity transformation on the photon field by

$$\begin{aligned} A^0_P(x) &\equiv \mathcal{P}A^0(x)\mathcal{P}^{-1} = A^0(\tilde{x})\,, \\ \boldsymbol{A}_P(x) &\equiv \mathcal{P}\boldsymbol{A}(x)\mathcal{P}^{-1} = -\boldsymbol{A}(\tilde{x})\,, \end{aligned} \tag{10.20}$$

or by

$$\begin{aligned} A^0_P(x) &\equiv \mathcal{P}A^0(x)\mathcal{P}^{-1} = -A^0(\tilde{x})\,, \\ \boldsymbol{A}_P(x) &\equiv \mathcal{P}\boldsymbol{A}(x)\mathcal{P}^{-1} = \boldsymbol{A}(\tilde{x})\,. \end{aligned} \tag{10.21}$$

From the point of view of the free field theory, we cannot choose between the two. However, once we introduce the sources and find solutions for the electromagnetic field in some specific cases, the situation changes. For example, if the source is a point charge q at rest at the origin, the solution for $\boldsymbol{E}$ is the Coulomb solution:

$$\boldsymbol{E} = \frac{q}{4\pi r^3}\boldsymbol{x}\,. \tag{10.22}$$

Since the right hand side is odd under parity transformation, we conclude that the electric field $\boldsymbol{E}$ must be odd under parity. This forces us to choose Eq. (10.20) as the correct parity transformation property of the photon field, because $E^i = \partial_0 A_i - \partial_i A_0$.

□ **Exercise 10.5** Show that the parity transformation rules defined in both Eqs. (10.20) and (10.21) above keep the free Maxwell equations (i.e., Maxwell equations in the absence of sources) invariant.

10.2.4 Interacting fields

We have thus shown that for all kinds of fields discussed earlier, the free Lagrangians obey parity invariance. Of course, free Lagrangians are no good for describing any physical phenomena, since physical phenomena are interactions. The question therefore is whether parity remains a good symmetry, i.e., Eq. (10.8) holds in the presence of interactions. The answer depends on what the interactions are. Here we will discuss a few examples to gain some experience in this matter.

Let us start with a theory of fermions and real scalars, with an interaction term

$$\mathscr{L}_{\rm int} = -h\bar{\psi}\psi\phi\,, \tag{10.23}$$

which was introduced in Eq. (5.9). Under parity transformation, the combination $\bar{\psi}\psi$ remains invariant, as is evident from Eq. (10.16). Thus the above interaction term is invariant only if we have $\eta_P = +1$ for the field ϕ. In other words, among the two choices available for parity transformation of free scalar fields, we are forced to accept only one.

If instead we have an interaction of the form

$$\mathscr{L}_{\rm int} = -h'\,\bar{\psi}\gamma_5\psi\phi\,. \tag{10.24}$$

parity invariance is obtained only if $\phi_P(x) = -\phi(\tilde{x})$ by the same arguments as above. Scalar fields with negative intrinsic parity or odd parity are called *pseudoscalar fields*. Using this terminology, we can rephrase the earlier statement by saying that in a theory where the term in Eq. (10.24) is the only interaction term, we have parity invariance in the theory provided the field ϕ is pseudoscalar.

If we now include the self-interaction terms of scalar fields in the interaction Lagrangian, we find that pseudoscalar fields can only have even order interaction terms, e.g., a ϕ^4-term, if parity is a symmetry of the Lagrangian. If we have a theory with the interaction

$$\mathscr{L}_{\rm int} = -h'\,\bar{\psi}\gamma_5\psi\phi - \mu\phi^3 - \lambda\phi^4\,. \tag{10.25}$$

we find that parity is not a symmetry of the interaction Lagrangian. This is because with $\phi_P(x) = -\phi(\tilde{x})$, the ϕ^3-term is not invariant under parity. On the other hand, if we choose $\phi_P(x) = \phi(\tilde{x})$, both ϕ^3 and ϕ^4 terms are invariant, but the interaction with the fermions

is not. Thus there is no definition of the intrinsic parity of the scalar field which makes this Lagrangian parity invariant.

Such an impasse can also occur without self-interactions of the scalar. For example, suppose we have a theory in which the interaction terms are of the form

$$\mathscr{L}_{\text{int}} = -\overline{\psi}(h + h'\gamma_5)\psi\phi\,. \tag{10.26}$$

No definition of intrinsic parity of ϕ can make this interaction parity invariant. This is because the term with γ_5 would demand the parity of the scalar field to be -1 in order to be parity invariant, whereas the other term would demand the same quantity to be $+1$ in order to be parity invariant.

□ **Exercise 10.6** In the absence of a mass term for the fermion, the third equation of Eq. (10.15) need not be satisfied.

a) Show that with this choice $\gamma_0\gamma_5$ is a solution, i.e., $\mathsf{P} = \eta'_P\gamma_0\gamma_5$ will satisfy the other two equations.

b) Show that in this case, the interaction in Eq. (10.23) implies that ϕ is a pseudoscalar, whereas that of Eq. (10.24) forces ϕ to be a scalar.

c) Show that even in this case, the interactions in Eq. (10.25) and Eq. (10.26) cannot be made parity-invariant by any choice of intrinsic parity of ϕ.

We now consider the Lagrangian of QED. Here, the interaction term is

$$\begin{aligned}\mathscr{L}_{\text{int}} &= -eQ\,\overline{\psi}\gamma_\mu\psi A^\mu \\ &= -eQ\left[\overline{\psi}\gamma_0\psi A_0 - \overline{\psi}\gamma_i\psi A_i\right]\,,\end{aligned} \tag{10.27}$$

as discussed in §9.1. For the fermion bilinears, we find

$$\overline{\psi}_P(x)\gamma_0\psi_P(x) = \Big(\eta_P\psi^\dagger(\tilde{x})\Big)\gamma_0\Big(\eta_P\gamma_0\psi(\tilde{x})\Big) = \overline{\psi}(\tilde{x})\gamma_0\psi(\tilde{x})\,, \tag{10.28}$$

and

$$\overline{\psi}_P(x)\gamma_i\psi_P(x) = \Big(\eta_P\psi^\dagger(\tilde{x})\Big)\gamma_i\Big(\eta_P\gamma_0\psi(\tilde{x})\Big) = -\,\overline{\psi}(\tilde{x})\gamma_i\psi(\tilde{x})\,. \tag{10.29}$$

Using these, we find that the interaction Lagrangian is in fact invariant under parity if the parity transformation rules for A_0 and A_i are given by Eq. (10.20). Fields transforming like the photon field under parity are called *vector fields*. If on the other hand there are spin-1 fields which transform like Eq. (10.21) under parity, they would be called *axial vector fields*.

□ **Exercise 10.7** Show that, if for some spin-1 field B_μ, the interaction with fermions is given by

$$\mathscr{L}_{\rm int} = a\,\overline{\psi}\gamma^\mu\gamma_5\psi B_\mu \tag{10.30}$$

where a is a constant, the Lagrangian is parity invariant provided B_μ is an axial vector field.

□ **Exercise 10.8** Show that, if for some spin-1 field Z_μ, the interaction with fermions is given by

$$\mathscr{L}_{\rm int} = \overline{\psi}\gamma^\mu(a + b\gamma_5)\psi Z_\mu \tag{10.31}$$

where a and b are constants, the Lagrangian cannot be parity invariant.

10.3 Charge conjugation

Charge conjugation was defined in §10.1 as the operation of changing the electric charge of any particle involved. In a more general setting where we will be discussing not only electromagnetic interactions but other types of interactions as well, the charge conjugation operation would mean changing the sign of not only the electric charge, but of all types of charges that a particle may possess. Since we know that any charge of a particle should be equal and opposite to that of the antiparticle, in effect this means interchanging particles and antiparticles.

10.3.1 Free fields

For scalar fields, this just means replacing ϕ by $\phi^\dagger$, and it is easy to see that the free Lagrangian of a scalar field is invariant under this operation. More generally, we can define the effect of the charge conjugation operation to be

$$\phi_C(x) \equiv C\phi(x)C^{-1} = \eta_C\phi^\dagger(x)\,, \tag{10.32}$$

where C is the charge conjugation operator, and η_C can be called the *intrinsic charge conjugation phase* of the field. And, of course, the free Lagrangian for a complex scalar field is invariant under this symmetry for any phase η_C. For real scalar fields, since both ϕ and ϕ_C have to be hermitian, ϕ should be an eigenstate of the charge conjugation operator with eigenvalue $+1$ or -1.

For the photon field which is also a real field, the free Lagrangian again demands that charge conjugation is a symmetry provided the intrinsic charge conjugation factor is either $+1$ or -1. When we bring in interactions, we find that the Maxwell equations are invariant only if we choose

$$A^{\mu}_{C}(x) \equiv C A^{\mu}(x) C^{-1} = -A^{\mu}(x) \tag{10.33}$$

under charge conjugation operation. This is because the current changes sign if all the charges are reversed, $Cj^{\mu}C^{-1} = -j^{\mu}$, and j^{μ} appears in the Lagrangian in the combination $j^{\mu}A_{\mu}$.

For the fermion fields, the charge conjugation operation is a bit more complicated since the fermion fields are spinor fields. We cannot just replace ψ by $\psi^{\dagger}$ in the free Dirac Lagrangian, since $\psi^{\dagger}$ is thought of as a row matrix whereas ψ is a column matrix. This problem can be easily avoided by considering, instead of $\psi^{\dagger}$, its transpose, which we can call ψ^{*}. But even this cannot replace ψ because ψ^{*} does not have the same Lorentz transformation property as ψ. This can be seen as follows. In §4.2, we showed that under a Lorentz transformation

$$x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu}\, x^{\nu}\,, \tag{10.34}$$

the spinor fields transform like

$$\psi'(x') = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x)\,. \tag{10.35}$$

Thus, $\psi^{*}(x)$ would transform like

$$\psi^{*\prime}(x') = \exp\left(+\frac{i}{4}\sigma^{*}_{\mu\nu}\omega^{\mu\nu}\right)\psi^{*}(x)\,. \tag{10.36}$$

This is not the same transformation since $\sigma^{*}_{\mu\nu} \neq -\sigma_{\mu\nu}$. Thus, if we were to make the simple-minded replacement of ψ by ψ^{*}, we would end up with a Lagrangian that is not Lorentz invariant.

This, however, is not an impasse. Even for the case of parity, ψ_P and ψ were not proportional. Rather they were related by a matrix P, which was defined in Eq. (10.13). In the same manner, here we can ask whether the charge conjugation transform of the field ψ, which will be called ψ_C, can be related to ψ^{*} by a matrix. In other words, we will look for a unitary matrix C such that ψ_C can be defined as

$$\psi_C \equiv C\gamma_0^{\mathsf{T}}\psi^{*}\,, \tag{10.37}$$

with the condition that ψ_C transforms the same way as ψ under Lorentz transformations. The appearance of γ_0 in this equation is purely conventional. We could have called the combination $C\gamma_0^T$ by some name C' and proceeded to look for C'.

The essential property of the desired matrix C was already suggested in Ex. 4.6 (p 54). We now derive this property for the reader who has not already solved that exercise. We want ψ_C, as defined in Eq. (10.37), to transform according to Eq. (10.35). This means

$$C\gamma_0^T \psi'^*(x') = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right) C\gamma_0^T \psi^*(x)\,. \tag{10.38}$$

We know from Eq. (10.36) how ψ^* transforms. So this can be rewritten as

$$C\gamma_0^T \exp\left(+\frac{i}{4}\sigma^*_{\mu\nu}\omega^{\mu\nu}\right)\psi^*(x) = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right) C\gamma_0^T\psi^*(x). \tag{10.39}$$

If this has to be satisfied for arbitrary $\omega_{\mu\nu}$ and arbitrary ψ, the matrix C must satisfy the relation

$$C\gamma_0^T\sigma^*_{\mu\nu} = -\sigma_{\mu\nu}C\gamma_0^T\,, \tag{10.40}$$

or

$$C^{-1}\sigma_{\mu\nu}C = -\gamma_0^T\sigma^*_{\mu\nu}\gamma_0^T = -\sigma^T_{\mu\nu}\,. \tag{10.41}$$

The last equation is obtained by using the definition of the σ-matrices and the hermiticity of the γ-matrices given in Eq. (4.18).

The question now is, does there exist a matrix C which satisfies Eq. (10.41) for all six of the matrices $\sigma_{\mu\nu}$? The first step towards an answer is to realize that if we can find a matrix C which satisfies the relation

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T\,, \tag{10.42}$$

this C will definitely satisfy Eq. (10.41), because

$$[\gamma_\mu,\gamma_\nu]^T_- = \left[\gamma_\nu^T,\gamma_\mu^T\right]_- = -\left[\gamma_\mu^T,\gamma_\nu^T\right]_-\,. \tag{10.43}$$

Secondly, if the matrices γ_μ satisfy the anticommutation relation of Eq. (4.9), the matrices $-\gamma_\mu^T$ also do so. In other words, since

$$[\gamma_\mu,\gamma_\nu]_+ = 2g_{\mu\nu}\,, \tag{10.44}$$

we can take a transpose of both sides of this equation and obtain

$$\left[-\gamma_\mu^\mathsf{T}, -\gamma_\nu^\mathsf{T}\right]_+ = 2g_{\mu\nu}\,. \tag{10.45}$$

By the theorem mentioned in connection with Eq. (4.20), we can therefore say that there must exist a unitary matrix C such that Eq. (10.42) is satisfied. The precise form of this matrix will depend on the representation of the γ-matrices.

□ **Exercise 10.9** Show that under charge conjugation the fermion current transforms as $\overline{\psi}\gamma_\mu\psi \to -\overline{\psi}\gamma_\mu\psi$. Show that the current for complex scalars change sign as well.

□ **Exercise 10.10** Show that

$$\mathsf{C}^{-1}\gamma_5\mathsf{C} = \gamma_5^\mathsf{T}\,. \tag{10.46}$$

□ **Exercise 10.11** Using $(\psi_C)_C = \psi$, show that the matrix C must be antisymmetric.

□ **Exercise 10.12** Consider two different representations of γ-matrices related by $\tilde{\gamma}_\mu = U\gamma_\mu U^\dagger$ for some unitary matrix U. Show that the charge conjugation matrices in the two representations are related by

$$\tilde{\mathsf{C}} = U\mathsf{C}U^\mathsf{T}\,. \tag{10.47}$$

□ **Exercise 10.13** Show that, in the Dirac-Pauli representation of the γ-matrices, a matrix C that satisfies Eq. (10.42) is

$$\mathsf{C} = i\gamma^2\gamma^0\,. \tag{10.48}$$

In the Majorana representation of γ-matrices given in Appendix A.1, show that one can define

$$\mathsf{C} = i\gamma^0\,. \tag{10.49}$$

□ **Exercise 10.14** Use the conjugation matrix defined in Eq. (10.48) to verify that our choice of the spinor solutions in Eqs. (4.61) and (4.62) satisfy the relations

$$\begin{aligned}\mathsf{C}\gamma_0^\mathsf{T}u_s^*(\boldsymbol{p}) &= v_s(\boldsymbol{p})\,,\\ \mathsf{C}\gamma_0^\mathsf{T}v_s^*(\boldsymbol{p}) &= u_s(\boldsymbol{p})\,.\end{aligned} \tag{10.50}$$

10.3.2 Interactions

Real scalar fields are invariant under charge conjugation, so any interactions containing only real scalar fields will of course be invariant under the same operation. For theories with complex scalar fields, the interactions are invariant if they contain only the combination $\phi^\dagger\phi$. Let us now check a non-trivial case, viz., that of the interactions involving fermion fields.

Because of Lorentz invariance the fermion fields always appear in pairs, in bilinears of the form

$$\overline{\psi}_1 F \psi_2 \tag{10.51}$$

with some constant matrix F sandwiched between them. We have kept the bilinear in a general form in which the fermion fields ψ_1 and ψ_2 need not be the same. In order to find the charge conjugation properties of fermion interactions, we need to know how such a bilinear behaves under charge conjugation.

The charge conjugated form for the bilinear above is

$$(\overline{\psi}_1)_C F (\psi_2)_C \,. \tag{10.52}$$

For any fermion field ψ,

$$\overline{\psi}_C = \psi_C^\dagger \gamma_0 = \psi^\mathsf{T} \gamma_0^* \mathsf{C}^{-1} \gamma_0 \,, \tag{10.53}$$

using $\mathsf{C}^\dagger = \mathsf{C}^{-1}$. Recalling now that $\gamma_0^* = \gamma_0^\mathsf{T}$ and using the definition of the matrix C from Eq. (10.42), we obtain

$$\overline{\psi}_C = -\,\psi^\mathsf{T} \mathsf{C}^{-1} \,. \tag{10.54}$$

Thus for the expression in Eq. (10.52), we can write

$$(\overline{\psi}_1)_C F (\psi_2)_C = -\,\psi_1^\mathsf{T} \mathsf{C}^{-1} F \mathsf{C} \gamma_0^\mathsf{T} \psi_2^* \,. \tag{10.55}$$

This looks very different from the bilinears that we have seen so far, where the field at the left has a bar on it. But it can be put into that form. To see that let us start with a bilinear of the form $\psi_1^\mathsf{T} M \gamma_0^\mathsf{T} \psi_2^*$, and rewrite it by explicitly putting the spinor indices:

$$\begin{aligned} \psi_1^\mathsf{T} M \gamma_0^\mathsf{T} \psi_2^* &= (\psi_1)_\alpha M_{\alpha\beta} \left(\gamma_0^\mathsf{T}\right)_{\beta\gamma} (\psi_2)_\gamma^* \\ &= -\,(\psi_2)_\gamma^* (\gamma_0)_{\gamma\beta} \left(M^\mathsf{T}\right)_{\beta\alpha} (\psi_1)_\alpha \,. \end{aligned} \tag{10.56}$$

In the last step, we have merely written the objects in a different order. In the process, we had to interchange the positions of two fermionic operators, which brought in a minus sign because of the anticommuting nature of the fermion fields. Now we can revert to the matrix notation to write the expression as

$$\psi_1^{\mathsf{T}} M \gamma_0^{\mathsf{T}} \psi_2^* = -\psi_2^\dagger \gamma_0 M^{\mathsf{T}} \psi_1 = -\overline{\psi}_2 M^{\mathsf{T}} \psi_1 \,. \tag{10.57}$$

This is an identity. Using it on Eq. (10.55), we can write

$$(\overline{\psi}_1)_C F (\psi_2)_C = \overline{\psi}_2 \left(\mathsf{C}^{-1} F \mathsf{C}\right)^{\mathsf{T}} \psi_1 \,. \tag{10.58}$$

We can now use it to find the charge conjugation properties of various fermion interactions. As an example, consider the interaction with a pseudoscalar field, given by

$$\mathscr{L}_{\rm int} = h\, \overline{\psi}_1 \gamma_5 \psi_2 \phi - h^*\, \overline{\psi}_2 \gamma_5 \psi_1 \phi^\dagger \,. \tag{10.59}$$

Notice that the second term is the hermitian conjugate of the first one, and both are necessary in order that the interaction Lagrangian is hermitian. If the charge conjugation phase of the scalar field is taken to be η_ϕ, the charge conjugated interaction would be

$$\begin{aligned} (\mathscr{L}_{\rm int})_C &= h\, (\overline{\psi}_1)_C \gamma_5 (\psi_2)_C\, \eta_\phi \phi^\dagger - h^*\, (\overline{\psi}_2)_C \gamma_5 (\psi_1)_C \eta_\phi^* \phi \\ &= h\, \eta_\phi \overline{\psi}_2 \gamma_5 \psi_1 \phi^\dagger - h^*\, \eta_\phi^* \overline{\psi}_1 \gamma_5 \psi_2 \phi \,, \end{aligned} \tag{10.60}$$

using Eq. (10.58) in the last step. We see that this is the same as the original interaction if $h\eta_\phi = -h^*$. Which means that even for a complex coupling constant h, this Lagrangian can be made charge conjugation invariant by appropriately choosing η_ϕ.

□ **Exercise 10.15** Show that the interaction term in QED is invariant under charge conjugation provided A^μ transforms as in Eq. (10.33).

10.4 Time reversal

10.4.1 Antilinearity

Time reversal symmetry was defined in Eq. (10.3) as the reversal of the arrow of time. Discussion of this symmetry is somewhat more

involved than the earlier ones. Let us explain why. Suppose we are considering a physical process where some initial state $|\Psi\rangle$ turns into some final state $|\Psi'\rangle$. In the time reversed picture, the final state would become the initial one, and vice versa. In other words, if we denote the time reversal operator by $\mathcal{T}$, time reversal symmetry would imply

$$\langle \mathcal{T}\Psi' | \mathcal{T}\Psi \rangle = \langle \Psi | \Psi' \rangle \ . \tag{10.61}$$

An operator $\mathcal{U}$ satisfying the relation

$$\langle \mathcal{U}\Psi' | \mathcal{U}\Psi \rangle = \langle \Psi' | \Psi \rangle \tag{10.62}$$

is called a *unitary* operator. In analogy, operators which satisfy Eq. (10.61) are called *anti-unitary*.

There is a very powerful theorem due to Wigner which states that any symmetry operation, i.e., any transformation on states of a Hilbert space which leaves probabilities of all physical processes invariant, can be represented either by a unitary operator which is linear, or by an anti-unitary operator which is antilinear. We will not prove the theorem here, but let us explain the words "linear" and "antilinear".

Consider two arbitrary states $|\Psi_1\rangle$ and $|\Psi_2\rangle$. Suppose an operator $\mathcal{O}$ has the property that

$$\mathcal{O}\left(a_1 |\Psi_1\rangle + a_2 |\Psi_2\rangle\right) = a_1 \mathcal{O} |\Psi_1\rangle + a_2 \mathcal{O} |\Psi_2\rangle \tag{10.63}$$

for all complex numbers a_1 and a_2. Then the operator is *linear*. If on the other hand the operator is such that

$$\mathcal{O}\left(a_1 |\Psi_1\rangle + a_2 |\Psi_2\rangle\right) = a_1^* \mathcal{O} |\Psi_1\rangle + a_2^* \mathcal{O} |\Psi_2\rangle \tag{10.64}$$

for all a_1 and a_2, the operator is *antilinear*. The time reversal operator, since it is anti-unitary, must be antilinear, as the theorem tells us.

10.4.2 Free fields

With this background, let us discuss whether the free Lagrangians of different kinds of fields are invariant under the time reversal operation. For spin-0 as well as the electromagnetic fields, there is

not much to discuss. The arguments are very similar to what we used for showing parity invariance. The time reversed system has co-ordinates

$$x_T \equiv (-t, \boldsymbol{x}) = -\tilde{x}\,, \tag{10.65}$$

where $\tilde{x}$ was introduced in Eq. (10.6). As in the case of parity, the intrinsic phase associated with time reversal can be arbitrary for both spin-0 and spin-1 fields as long as we consider the free Lagrangian only. For the photon field, the interactions dictate that under time reversal one should have

$$\mathcal{T} A^0(x) \mathcal{T}^{-1} = A^0(-\tilde{x})\,, \qquad \mathcal{T} \boldsymbol{A}(x) \mathcal{T}^{-1} = -\boldsymbol{A}(-\tilde{x})\,. \tag{10.66}$$

This is because under time reversal,

$$\mathcal{T} \boldsymbol{j}(x) \mathcal{T}^{-1} = -\boldsymbol{j}(-\tilde{x})\,, \qquad \mathcal{T} j^0(x) \mathcal{T}^{-1} = j^0(-\tilde{x})\,, \tag{10.67}$$

since $\boldsymbol{j}$ represents the flow of charge whereas j^0 represents the static charge density.

In order to discuss time reversal properties of Dirac fields, we write the free Lagrangian in the form

$$\mathscr{L}(x) = \psi^\dagger(x) \left(i\partial_t + i\gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} - m\gamma_0\right) \psi(x)\,. \tag{10.68}$$

The time reversed form of the Lagrangian is

$$\begin{aligned} \mathcal{T} \mathscr{L}(x) \mathcal{T}^{-1} &= \psi_T^\dagger(x) \left(-i\partial_t - i\gamma_0^* \boldsymbol{\gamma}^* \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} - m\gamma_0^*\right) \psi_T(x) \\ &= \psi_T^\dagger(x) \left(-i\partial_t + i\gamma_0^{\mathsf{T}} \boldsymbol{\gamma}^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} - m\gamma_0^{\mathsf{T}}\right) \psi_T(x)\,. \end{aligned} \tag{10.69}$$

The complex conjugates of the γ-matrices, as well as $-i$, have appeared in the first step because of the antilinearity of the operation. In the following step, we have used the hermiticity properties of the γ-matrices to turn the complex conjugates into transposes. In keeping with the notations used for parity and charge conjugation, we will define

$$\psi_T(x) \equiv \mathcal{T} \psi(x) \mathcal{T}^{-1} = \mathsf{T} \psi(-\tilde{x}) \tag{10.70}$$

for some matrix T which is independent of space-time. Then we can write Eq. (10.69) as

$$\mathcal{T} \mathscr{L}(x) \mathcal{T}^{-1} = \psi^\dagger(-\tilde{x}) \mathsf{T}^\dagger \left(i\partial_{-t} + i\gamma_0^{\mathsf{T}} \boldsymbol{\gamma}^{\mathsf{T}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} - m\gamma_0^{\mathsf{T}}\right) \mathsf{T} \psi(-\tilde{x})\,. \tag{10.71}$$

Time reversal invariance will require

$$\mathcal{T}\mathscr{L}(x)\mathcal{T}^{-1} = \mathscr{L}(-\tilde{x})\,. \tag{10.72}$$

This can be achieved if the matrix T satisfies the relations

$$\begin{aligned} \mathsf{T}^\dagger\mathsf{T} &= 1\,, \\ \mathsf{T}^\dagger\gamma_0^{\mathsf{T}}\gamma_i^{\mathsf{T}}\mathsf{T} &= \gamma_0\gamma_i\,, \\ \mathsf{T}^\dagger\gamma_0^{\mathsf{T}}\mathsf{T} &= \gamma_0\,. \end{aligned} \tag{10.73}$$

The first of these equations says that T has to be a unitary matrix — both unitary and anti-unitary operators are represented by unitary matrices. In the other two equations, we replace the transposes by Eq. (10.42) which defines the matrix C. The last equation then gives

$$\mathsf{T}\gamma_0\mathsf{T}^\dagger = \gamma_0^{\mathsf{T}} = -\,\mathsf{C}^{-1}\gamma_0\mathsf{C}\,. \tag{10.74}$$

This equation can be rewritten as

$$[\mathsf{CT}, \gamma_0]_+ = 0\,. \tag{10.75}$$

Using this, the second equation of Eq. (10.73) can be cast into the form

$$[\mathsf{CT}, \gamma_i]_+ = 0\,. \tag{10.76}$$

Thus the matrix CT anticommutes with all the γ-matrices. Therefore it has to be a multiple of γ_5. The most general solution for T is then

$$\mathsf{T} = \eta_T\mathsf{C}^{-1}\gamma_5\,, \tag{10.77}$$

where η_T is a phase factor. Thus, the time reversal property of a Dirac field is given by

$$\psi_T(x) = \mathcal{T}\psi(x)\mathcal{T}^{-1} = \eta_T\mathsf{C}^{-1}\gamma_5\psi(-\tilde{x})\,. \tag{10.78}$$

□ **Exercise 10.16** Consider two different representations of the γ-matrices related by $\tilde{\gamma}_\mu = U\gamma_\mu U^\dagger$. Show that the time-reversed Dirac fields in the two representations are related by the equation

$$\tilde{\psi}_T(x) = U^*\psi_T(x)\,. \tag{10.79}$$

10.4.3 Interactions

Since the free Lagrangians are invariant under time-reversal, we can now ask whether the interactions are. The task is easy if we have only scalar and spin-1 fields. For spinor fields which always appear as bilinears, we can start addressing this question by finding out how a bilinear behaves under time reversal.

Consider then a general form of a bilinear given by $h\,\overline{\psi}_1 F \psi_2$, where ψ_1 and ψ_2 are in general two different spinor fields, h is a coupling constant and F is a constant 4×4 matrix. Under time reversal, this becomes

$$\begin{aligned} \mathcal{T}\left(h\,\overline{\psi}_1(x) F \psi_2(x)\right)\mathcal{T}^{-1} &= h^*\mathcal{T}\left(\psi_1^\dagger(x)\gamma_0 F \psi_2(x)\right)\mathcal{T}^{-1} \\ &= h^*\,\psi_1^\dagger(-\tilde{x})\mathsf{T}^\dagger\gamma_0^* F^* \mathsf{T}\psi_2(-\tilde{x}) \\ &= h^*\,\overline{\psi}_1(-\tilde{x})\gamma_0\gamma_5 \mathsf{C}\gamma_0^{\mathsf{T}} F^* \mathsf{C}^{-1}\gamma_5\psi_2(-\tilde{x})\,. \end{aligned} \tag{10.80}$$

Using the definition of the matrix C from Eq. (10.42) to write $\mathsf{C}\gamma_0^{\mathsf{T}} = -\gamma_0\mathsf{C}$, we can put it in the form

$$\mathcal{T}\left(h\,\overline{\psi}_1(x) F \psi_2(x)\right)\mathcal{T}^{-1} = h^*\,\overline{\psi}_1(-\tilde{x}) F_{\mathcal{T}} \psi_2(-\tilde{x})\,, \tag{10.81}$$

where

$$F_{\mathcal{T}} = \gamma_5 \mathsf{C} F^* \mathsf{C}^{-1}\gamma_5\,. \tag{10.82}$$

We now list the form for $F_{\mathcal{T}}$ for various possibilities for F:

F	1	γ_5	γ_0	γ_i	$\gamma_0\gamma_5$	$\gamma_i\gamma_5$
$F_{\mathcal{T}}$	1	γ_5	γ_0	$-\gamma_i$	$\gamma_0\gamma_5$	$-\gamma_i\gamma_5$

(10.83)

Using this table, we can now discuss whether specific interactions involving fermions are time-reversal invariant. As an example, let us take the interaction Lagrangian of QED, which was given in Eq. (9.18). For our purpose, we write it as

$$\mathscr{L}_{\text{int}} = -eQ\left[\overline{\psi}\gamma_0\psi A_0 - \overline{\psi}\gamma_i\psi A_i\right]\,. \tag{10.84}$$

The charge eQ is a real number, so complex conjugation does not affect it. For the field operators, we can use the table in Eq. (10.83) and the transformation properties of the photon field from Eq. (10.66) to see that this interaction is indeed invariant under time-reversal.

□ **Exercise 10.17** Consider the interaction introduced in Eq. (10.31). Show that this interaction is invariant under time-reversal provided Z_μ transforms the same way under time-reversal as the photon field. [Hint: First show from the hermiticity argument that a and b have to be real.]

10.5 CP

We have already discussed the discrete operations C, P and T separately. Sometimes it is useful to talk about combinations of these operations. One such combination which is very useful is CP, i.e., the product of charge conjugation and parity.

The transformation of different fields under CP need not be discussed in detail. They can be obtained by combining the results of P and C operations, both of which have been discussed earlier. For example, for a fermion field, we would obtain

$$\begin{aligned} \mathcal{CP}\ \psi(x)\ (\mathcal{CP})^{-1} &= \mathcal{CP}\ \psi(x)\ \mathcal{P}^{-1}\mathcal{C}^{-1} = \mathcal{C}\eta_P\gamma_0\psi(\tilde{x})\mathcal{C}^{-1} \\ &= \eta_P\gamma_0\mathcal{C}\psi(\tilde{x})\mathcal{C}^{-1} = \eta_P\eta_C\gamma_0\mathsf{C}\gamma_0^{\mathsf{T}}\psi^*(\tilde{x})\,. \end{aligned} \tag{10.85}$$

Using the fundamental property of the matrix C from Eq. (10.42), we can rewrite this as

$$\mathcal{CP}\ \psi(x)\ (\mathcal{CP})^{-1} = -\eta_{CP}\mathsf{C}\psi^*(\tilde{x})\,, \tag{10.86}$$

where η_{CP} is an intrinsic CP-phase of the field, defined by

$$\eta_{CP} = \eta_P\eta_C\,. \tag{10.87}$$

□ **Exercise 10.18** Use the CP-transformation rule for a fermion field to show that, if two fermion fields ψ_1 and ψ_2 have the same intrinsic CP phase, the general fermion bilinear constructed from them transforms the following way under CP:

$$\begin{aligned} \mathcal{CP}\ \overline{\psi}_1(x)F\psi_2(x)\ (\mathcal{CP})^{-1} &= \psi_1^{\mathsf{T}}(\tilde{x})\mathsf{C}^{-1}\gamma_0F\mathsf{C}\psi_2^*(\tilde{x}) \\ &= \overline{\psi}_2(\tilde{x})\gamma_0\mathsf{C}F^{\mathsf{T}}\mathsf{C}^{-1}\gamma_0\psi_1(\tilde{x})\,, \end{aligned} \tag{10.88}$$

using Eq. (10.58) in the last step.

□ **Exercise 10.19** Suppose we define F_{CP} by the relation

$$\mathcal{CP}\ \overline{\psi}_1(x)F\psi_2(x)\ (\mathcal{CP})^{-1} = \overline{\psi}_2(\tilde{x})F_{CP}\psi_1(\tilde{x})\,. \tag{10.89}$$

Using the results of the previous problem, verify the following table giving F_{CP} corresponding various possibilities for F:

F	1	γ_5	γ_0	γ_i	$\gamma_0\gamma_5$	$\gamma_i\gamma_5$
F_{CP}	1	$-\gamma_5$	$-\gamma_0$	γ_i	$-\gamma_0\gamma_5$	$\gamma_i\gamma_5$

(10.90)

For the photon field, using the transformation properties under parity and charge conjugation given in Eqs. (10.20) and (10.33), we can write

$$\mathcal{CP}A^0(x)(\mathcal{CP})^{-1} = -A^0(\tilde{x}), \quad \mathcal{CP}\boldsymbol{A}(x)(\mathcal{CP})^{-1} = \boldsymbol{A}(\tilde{x}). \quad (10.91)$$

With this and Eq. (10.88), we can show that the QED interaction is CP invariant. But it is somewhat of an unnecessary exercise since we have already known that the QED Lagrangian is invariant separately under C and P.

The more interesting point to note is that if there is a spin-1 field which has both vector and axial vector type interactions with fermions, as for example was given in Eq. (10.31), the interaction is not separately invariant under C and P, but is invariant under the combined operator CP.

10.6 CPT

We now consider the product of all the three independent discrete symmetries discussed so far. For the sake of convenience, we use the shorthand

$$\Theta \equiv \mathcal{CPT}\,. \quad (10.92)$$

In this case, let us first see how the photon field transforms under this operation. Using the transformation property of the photon field under C, P and T separately, we obtain

$$\Theta A^\mu(x)\Theta^{-1} = -A^\mu(-x)\,. \quad (10.93)$$

Moving over now to Dirac fields, we find

$$\begin{aligned}\Theta\,\psi(x)\,\Theta^{-1} &= \mathcal{CPT}\,\psi(x)\,\mathcal{T}^{-1}\,\mathcal{P}^{-1}\mathcal{C}^{-1} \\ &= \eta_T \mathcal{C}^{-1}\gamma_5\,\mathcal{CP}\,\psi(-\tilde{x})\,\mathcal{P}^{-1}\mathcal{C}^{-1} \\ &= -\,\eta_T\eta_{CP}\mathcal{C}^{-1}\gamma_5\mathcal{C}\psi^*(-x)\,, \end{aligned} \quad (10.94)$$

using the CP transformation property from Eq. (10.86) in the last step. Using the definition of the matrix C, this can be rewritten as

$$\Theta\,\psi(x)\,\Theta^{-1} = -\,\eta_\Theta\gamma_5^{\mathsf{T}}\psi^*(-x)\,, \quad (10.95)$$

where

$$\eta_\Theta \equiv \eta_{CP}\eta_T = \eta_C\eta_P\eta_T \,. \tag{10.96}$$

If we change the order of the three operators in the definition of Θ, the phases for operators appearing to the right of $\mathcal{T}$ will have to be replaced by their complex conjugates since $\mathcal{T}$ acts antilinearly. Of course, that changes only the definition of η_Θ, but not any of the conclusions below.

For bilinears of the general form $\overline{\psi}_1 F \psi_2$, the CPT transformation property can be deduced from here. For this, we will have to remember that since time reversal is an antilinear operation whereas parity and charge conjugation are linear ones, the product CPT is antilinear. Therefore,

$$\begin{aligned}\Theta\,\overline{\psi}(x)\,\Theta^{-1} &= \Theta\,\psi(x)^\dagger\gamma_0\,\Theta^{-1} = \left(\Theta\,\psi(x)\,\Theta^{-1}\right)^\dagger\gamma_0^* \\ &= -\eta_\Theta^*\psi^{\mathsf{T}}(-x)\gamma_5^{\mathsf{T}}\gamma_0^{\mathsf{T}}\,,\end{aligned} \tag{10.97}$$

using the fact that γ_0 and γ_5 are hermitian matrices. Assuming for the moment that we can assign the same CPT-phase to all fermions, we obtain

$$\begin{aligned}\Theta\,\overline{\psi}_1(x)F\psi_2(x)\,\Theta^{-1} &= \psi_1^{\mathsf{T}}(-x)\gamma_5^{\mathsf{T}}\gamma_0^{\mathsf{T}}F^*\gamma_5^{\mathsf{T}}\psi_2^*(-x) \\ &= \overline{\psi}_2(-x)\gamma_5\gamma_0F^\dagger\gamma_0\gamma_5\psi_1(-x)\,,\end{aligned} \tag{10.98}$$

using the identity of Eq. (10.57) at the last step. Let us write this as

$$\Theta\,\overline{\psi}_1(x)F\psi_2(x)\,\Theta^{-1} = \overline{\psi}_2(-x)F_\Theta\psi_1(-x)\,, \tag{10.99}$$

where

$$F_\Theta = \gamma_5\gamma_0F^\dagger\gamma_0\gamma_5 = \gamma_5F^\dagger\gamma_5\,, \tag{10.100}$$

using the notation $F^\dagger = \gamma_0F^\dagger\gamma_0$ that was introduced in Eq. (7.15). The significance of this symbol is that the hermitian conjugate of the general bilinear discussed above is given by

$$\left(\overline{\psi}_1F\psi_2\right)^\dagger = \overline{\psi}_2F^\dagger\psi_1\,. \tag{10.101}$$

If there is a term in the Lagrangian containing the bilinear $\overline{\psi}_1F\psi_2$, there must also be a term containing the bilinear $\overline{\psi}_2F^\dagger\psi_1$ since the

Lagrangian is hermitian. In fact, the CPT-conjugate of any term involving $\overline{\psi}_1 F \psi_2$ must equal the term involving $\overline{\psi}_2 F^\dagger \psi_1$ if the Lagrangian has to be CPT invariant.

Since γ_5 anticommutes with γ_μ, we find from Eq. (10.100) that if we assign the same intrinsic CPT phase for all fermions, $F^\dagger$ and F_Θ are equal for all fermion bilinears involving an even number of vector indices. On the other hand, for fermion bilinears carrying an odd number of vector indices, $F^\dagger$ and F_Θ differ by a sign.

The observation can be extended by looking back at the CPT transformation property of the photon field. It carries one vector index, and it differs from its CPT conjugate by a sign. Let us now carry it further and say that for any spin-1 field, we will define the intrinsic CPT phase to be the same as that of the photon field, i.e., any spin-1 field will transform under CPT as in Eq. (10.93). Moreover, for all spin-0 fields, we will define the intrinsic CPT phase so that

$$\Theta \phi(x) \Theta^{-1} = \phi^\dagger(-x)\,. \tag{10.102}$$

and will also take the CPT phase of all fermion fields to be $+1$. In that case, we can extend the statement made a little earlier. For these assignments of the intrinsic CPT phases of the fields, the CPT transformation property of any fermion bilinear or tensor (which includes scalar and vector) field carrying n number of Lorentz indices is related to the hermitian conjugate by a factor $(-1)^n$.

Since any term in the Lagrangian must be Lorentz invariant, it must have an even number of Lorentz indices, suitably contracted. So the CPT transform of any term would be the same as its complex (hermitian) conjugate, and we conclude that CPT is a good symmetry of the Lagrangian. Our argument makes no reference to whether we are talking of a free Lagrangian or one with interactions.

What have we assumed in arriving at this strong result? First, we have assumed that the Lagrangian is Lorentz invariant. There is an implicit second assumption, viz. that the spinor fields anticommute, a fact that we used in deriving Eq. (10.57), which has played a crucial role in the derivation of the transformation of fermion bilinears under CPT. Similarly, we have also assumed that fields with integral spin are quantized by commutation relations. Thus in any theory with these constraints, CPT must be a good symmetry of the Lagrangian. This is the *CPT theorem*. Although we have demonstrated it with

fields of spin 0, $\frac{1}{2}$ and 1 only, it is thought to be valid for fields with higher spins as well.

Chapter 11

Electromagnetic form factors

11.1 General electromagnetic vertex

Suppose we want the matrix element of the electromagnetic current operator between two physical or on-shell states of a fermion of charge eQ. Using the expression for the electromagnetic current from Eq. (9.13), we obtain

$$\begin{aligned}\langle \boldsymbol{p}', s' \left| j_\mu(x) \right| \boldsymbol{p}, s \rangle &= eQ \left\langle \boldsymbol{p}', s' \left| \overline{\psi}(x) \gamma_\mu \psi(x) \right| \boldsymbol{p}, s \right\rangle \\ &= \frac{e^{-iq\cdot x}}{\sqrt{2E_p V}\,\sqrt{2E_{p'} V}}\, \overline{u}_{s'}(\boldsymbol{p}')\, eQ\gamma_\mu\, u_s(\boldsymbol{p})\,. \quad (11.1)\end{aligned}$$

Here we have used the free field normalized states of §6.2, and written

$$q = p - p'\,, \quad (11.2)$$

which stands for the transferred momentum.

However, when we include interactions, we can no longer use the free fields to describe the current. Let us find out the general form of the matrix element of j_μ in this case. Since the momentum operator $\mathscr{P}_\mu$ generates space-time translations, we can write

$$j_\mu(x) = e^{i\mathscr{P}\cdot x} j_\mu(0) e^{-i\mathscr{P}\cdot x}\,. \quad (11.3)$$

So

$$\langle \boldsymbol{p}', s' \left| j_\mu(x) \right| \boldsymbol{p}, s \rangle = e^{-iq\cdot x} \langle \boldsymbol{p}', s' \left| j_\mu(0) \right| \boldsymbol{p}, s \rangle\,. \quad (11.4)$$

The matrix element of $j_\mu(0)$ is parametrized such that

$$\langle \boldsymbol{p}', s' \left| j_\mu(x) \right| \boldsymbol{p}, s \rangle = \frac{e^{-iq\cdot x}}{\sqrt{2E_p V}\,\sqrt{2E_{p'} V}}\, \overline{u}_{s'}(\boldsymbol{p}')\, e\Gamma_\mu(p, p')\, u_s(\boldsymbol{p}). \quad (11.5)$$

We have kept a factor of e outside, since this is a constant which will appear in any electromagnetic vertex. All the rest, which might depend on the specific particle states used here, have been dumped into an object called Γ_μ. We can think of this object as appearing from an effective interaction term $-\langle j_\mu \rangle A^\mu$ in the Lagrangian. Then the corresponding Feynman rule for the effective vertex is $-ie\Gamma_\mu$. For this reason, Γ_μ is called the *vertex function*, and we will now investigate some of its general features.

First of all, Γ_μ must contain only one free vector index. This means that the most general form for Γ_μ is

$$\begin{aligned}\Gamma_\mu = {} & \gamma_\mu(F_1 + \tilde{F}_1\gamma_5) + (iF_2 + \tilde{F}_2\gamma_5)\sigma_{\mu\nu}q^\nu \\ & + \tilde{F}_3 q_\mu \not{q}\gamma_5 + q_\mu(F_4 + \tilde{F}_4\gamma_5)\,. \end{aligned} \tag{11.6}$$

The vertex function does not contain terms dependent on $(p+p')_\mu$ because the matrix element of $(p+p')_\mu$ can be expressed in terms of those of γ_μ and $\sigma_{\mu\nu}q^\nu$. This is the result of Gordon identity, which was shown in Eq. (4.52). Also, notice that although the terms seem to come in pairs, one with a γ_5 factor and one without, there is no such pair for the $\tilde{F}_3$ term. This is because $\bar{u}(\boldsymbol{p}')\not{q}u(\boldsymbol{p}) = 0$.

The above form given for Γ_μ can be simplified by noting that the electromagnetic current is conserved. This is a consequence of Noether's theorem, which gives

$$0 = \langle \boldsymbol{p}', s' \left| \partial^\mu j_\mu(x) \right| \boldsymbol{p}, s \rangle = -iq^\mu \langle \boldsymbol{p}', s' \left| j_\mu(x) \right| \boldsymbol{p}, s \rangle \,. \tag{11.7}$$

Using Eq. (11.5), we can write this as

$$q^\mu\, \bar{u}_{s'}(\boldsymbol{p}')\, \Gamma_\mu\, u_s(\boldsymbol{p}) = 0\,. \tag{11.8}$$

Remembering that $\bar{u}(\boldsymbol{p}')\not{q}u(\boldsymbol{p}) = 0$, this gives the following condition:

$$\bar{u}_{s'}(\boldsymbol{p}') \left[(\tilde{F}_1 + \tilde{F}_3 q^2)\not{q}\gamma_5 + q^2(F_4 + \tilde{F}_4\gamma_5) \right] u_s(\boldsymbol{p}) = 0\,. \tag{11.9}$$

Since this condition has to hold for arbitrary values of $\boldsymbol{p}$ and $\boldsymbol{p}'$, we conclude that

$$\tilde{F}_1 = -\tilde{F}_3 q^2\,, \qquad F_4 = \tilde{F}_4 = 0\,. \tag{11.10}$$

Thus the most general parametrization consistent with gauge invariance is given by

$$\Gamma_\mu = \gamma_\mu F_1 + (iF_2 + \tilde{F}_2\gamma_5)\sigma_{\mu\nu}q^\nu + \tilde{F}_3(q_\mu \not{q} - q^2\gamma_\mu)\gamma_5\,. \tag{11.11}$$

The objects F_1, F_2, $\tilde{F}_2$ and $\tilde{F}_3$ appearing in this expression are called *form factors*.

Let us now discuss what these form factors can depend on. They are Lorentz invariant quantities, so they can depend only on the dot products of various 4-vectors in the problem. There are really only two independent 4-vectors, p and p'. With these we can construct three dot products, p^2, p'^2 and $p \cdot p'$. The first two of these are not variables, they just give the mass of the physical particle whose vertex we are examining. Thus we are left with only one variable which is Lorentz invariant. Instead of taking it to be $p \cdot p'$, we can also choose it to be $q^2 = (p - p')^2$. The form factors appearing in Eq. (11.11) are to be understood to be functions of q^2.

□ **Exercise 11.1** Consider the matrix element of the electromagnetic current between an initial state of momentum p and a final one of momentum p' of a complex scalar field. Using Lorentz invariance and current conservation, show that the most general vertex is of the form

$$(p + p')_\mu \, eF(q^2) , \tag{11.12}$$

where $F(q^2)$ is a form factor.

11.2 Physical interpretation of form factors

In some limits, the form factors defined above correspond to concepts such as charge and dipole moments, with which we are familiar from classical electrodynamics. Here we examine the physics of these form factors.

11.2.1 Charge form factor F_1

To take the first example, let us consider the situation where $\boldsymbol{p} = \boldsymbol{p}'$ and $s = s'$, i.e., the initial and the final states of the fermion are the same. The momentum transfer 4-vector q vanishes in this case, and therefore $q^2 = 0$. One now obtains

$$\begin{aligned} \langle \boldsymbol{p}, s \, | j_\mu(x) | \, \boldsymbol{p}, s \rangle &= \frac{eF_1(0)}{2E_p V} \, \bar{u}_s(\boldsymbol{p}) \, \gamma_\mu \, u_s(\boldsymbol{p}) \\ &= \frac{eF_1(0) p^\mu}{E_p V} . \end{aligned} \tag{11.13}$$

In deducing the last step, we have used the normalization of the spinors from Eq. (4.51). We have also used Gordon identity, whose restricted form was given in Eq. (4.50).

This expectation value is like a classical current. Let us now suppose that this current couples to an electromagnetic field for which $\boldsymbol{A} = 0$, and only $A^0 \equiv \varphi$ is non-vanishing. This could arise from the following *effective* interaction term in the Lagrangian:

$$-\langle j_\mu \rangle A^\mu = -\frac{eF_1(0)\varphi}{V}\,. \tag{11.14}$$

using $p^0 = E_p$. Since $\boldsymbol{A} = 0$, we expect the electromagnetic field in this case to be purely electric, and the Lagrangian term to be $-\rho\varphi$, where ρ is the charge density. Thus here $\rho = eF_1(0)/V$, and since this is the result of the calculation in a volume V, we conclude that

$$eF_1(0) = \text{charge of the particle.} \tag{11.15}$$

Indeed, this agrees with the tree-level result, for which $F_1 = Q$, as can be seen from Eq. (11.1). In other words, $F_1(0)$ is the charge, in the fundamental unit of e, of the particle whose vertex is being considered. For this reason, $F_1(q^2)$ is often called the *charge form factor*.

Let us consider a more general form for the photon field A^μ. The F_1-term in the matrix element of Eq. (11.5) can be rewritten as

$$\frac{e^{-iq\cdot x}}{\sqrt{2E_pV}\,\sqrt{2E_{p'}V}}\,\frac{eF_1(q^2)}{2m}\overline{u}_{s'}(\boldsymbol{p}')\left[(p+p')_\mu - i\sigma_{\mu\nu}q^\nu\right]u_s(\boldsymbol{p}), \tag{11.16}$$

using the Gordon identity given in Eq. (4.52). Now consider the non-relativistic case, i.e., when $E_p \approx E_{p'} \approx m$. The dominant contribution from the $(p+p')_\mu$ in the square brackets reduces to the electric charge, which has been discussed above. So let us now concentrate on the other term. In the Lagrangian, the contribution of this term can be obtained if we couple this term to a photon field A^μ. The resultant effective term is

$$-\langle j_\mu \rangle A^\mu = \frac{e^{-iq\cdot x}}{2mV}\,\frac{eF_1(q^2)}{2m}\overline{u}_{s'}(\boldsymbol{p}')i\sigma_{\mu\nu}q^\nu u_s(\boldsymbol{p})A^\mu(q)\,. \tag{11.17}$$

In the convention that q_0 is positive, the photon is being created at the vertex, so in the Fourier expansion for the photon field, it is the

A^{μ}_{-} term, i.e., the term containing the factor $e^{iq\cdot x}$, which is relevant here. With this term, the Fourier transform of $F^{\mu\nu}$ would read

$$F^{\mu\nu}(q) = i\Big(q^{\mu}A^{\nu}(q) - q^{\nu}A^{\mu}(q)\Big)\,. \tag{11.18}$$

Noting the antisymmetry of $\sigma_{\mu\nu}$, we can thus rewrite the expression of Eq. (11.17) in the following form:

$$-\langle j_{\mu}\rangle A^{\mu} = -\frac{e^{-iq\cdot x}}{2mV}\,\frac{eF_1(q^2)}{4m}\overline{u}_{s'}(\boldsymbol{p}')\sigma_{\mu\nu}u_s(\boldsymbol{p})F^{\mu\nu}(q)\,. \tag{11.19}$$

At this point, it is instructive to look at the Dirac-Pauli representation of the γ-matrices to gain further insight. In the non-relativistic limit, the solution of the u-spinors given in Eq. (4.61) have negligible lower components, and upper components that are just χ_s defined in Eq. (4.60). In other words, in this representation we can write

$$u_s(\boldsymbol{p}) = \sqrt{2m}\begin{pmatrix}\chi_s\\ 0\end{pmatrix} + \text{small terms}\,, \tag{11.20}$$

using the normalization of Ch. 4 and using $E \approx m$. This form of the spinor solutions shows that in this representation, the dominant contribution to the spinor bilinear given above will come from the components of $\sigma_{\mu\nu}$ which have a non-zero entry in the upper-left corner. These are

$$\sigma_{ij} = \epsilon_{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\,. \tag{11.21}$$

whereas σ_{0i}'s have a zero in the upper-left corner. Putting this into Eq. (11.19), we obtain the dominant contribution to the matrix element to be

$$-\frac{e^{-iq\cdot x}}{V}\,\frac{eF_1(q^2)}{4m}\chi^{\dagger}_{s'}\sigma^k\chi_s\epsilon_{ijk}F^{ij}(q)\,. \tag{11.22}$$

Looking back at the tensor $F^{\mu\nu}$ in Eq. (8.6), we can see that $\epsilon_{ijk}F^{ij}(q) = 2B_k(q)$, so that the last expression can be written as

$$\frac{e^{-iq\cdot x}}{V}\,\frac{eF_1(q^2)}{2m}\chi^{\dagger}_{s'}\boldsymbol{\sigma}\cdot\boldsymbol{B}\chi_s\,, \tag{11.23}$$

using the fact that $\sigma^k B_k = -\sigma_k B_k = -\boldsymbol{\sigma}\cdot\boldsymbol{B}$.

Since we do not know the explicit form of the form factor $F_1(q^2)$, it is impossible to tell how this interaction should look like in the co-ordinate space. However, let us expand $F_1(q^2)$ in a Taylor series around $q^2 = 0$. Then the first term is a constant, and at least for this term it is easy to find the inverse Fourier transform of the expression in Eq. (11.23). In fact, inverse Fourier transform would be the matrix element of the operator

$$\frac{1}{V}\,\frac{eF_1(0)}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B} = \frac{1}{V}\,\frac{eQ}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B} \tag{11.24}$$

between $\chi^\dagger_{s'}$ and χ_s, where now $\boldsymbol{B}$ is the magnetic field in co-ordinate space. Once again, remember that this calculation is done within a volume V. For the total Lagrangian, we need to multiply by V, assuming a constant $\boldsymbol{B}$-field. The corresponding term in the total Hamiltonian will differ by a sign, i.e., will be

$$-\frac{eQ}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\,. \tag{11.25}$$

This is exactly the interaction of a particle with a magnetic moment

$$\boldsymbol{\mu}_D = \frac{eQ}{2m}\boldsymbol{\sigma} = \frac{eQ}{m}\boldsymbol{S}\,, \tag{11.26}$$

where $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\sigma}$ is the spin vector for the particle. We have put a subscript D on the magnetic moment since this contribution to the magnetic moment, coming from the charge form factor, is usually called the *Dirac magnetic moment.*

Usually, the magnetic moment is expressed in terms of the Landé g-factor, which is defined by

$$\boldsymbol{\mu} = \frac{eQ}{2m}g\,\boldsymbol{S}\,, \tag{11.27}$$

where eQ is the charge of the particle and m its mass. Comparing, we see that the charge form factor gives the following contribution to the g-factor:

$$g_D = 2\,. \tag{11.28}$$

11.2.2 Anomalous magnetic moment F_2

Let us now move on to the interpretation of the form factor F_2. The task is easy here, since this term looks exactly like the term we had been discussing so far. Going through the same steps, we can now conclude that there is another contribution to the magnetic moment coming from the form factor F_2. This is called the *anomalous magnetic moment*, which we will denote by μ_A. It is given by

$$\mu_A = -eF_2(0)\sigma = -2eF_2(0)S\,. \tag{11.29}$$

For a particle with a charge eQ, the Landé g-factor is thus obtained by summing the two contributions:

$$g = 2 - \frac{4m}{Q}F_2(0)\,. \tag{11.30}$$

But the interesting point is that even if a particle does not have a charge, there is no fundamental reason why it cannot have a non-zero value of the form factor F_2. Thus even uncharged particles may have magnetic moments. For such particles, of course, the entire magnetic moment is anomalous.

11.2.3 Electric dipole moment $\tilde{F}_2$

The analysis of this form factor is very similar to that of the form factor F_2. The only difference is that there is now an extra γ_5. Thus when we go to the Dirac-Pauli representation, we now will have to ask which of the matrices $\sigma_{\mu\nu}\gamma_5$ have non-zero entries in the upper-left corner. These are

$$\sigma_{0k}\gamma_5 = -i\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\,. \tag{11.31}$$

So now, instead of the components F^{ij} of the field tensor, we have the components F^{0i} which give the dominant contribution. And these are the components of the electric field. Thus instead of $\boldsymbol{B}$ in the effective interaction, we now will have $\boldsymbol{E}$. The interaction is therefore that of an electric dipole interacting with the electric field, and the dipole moment is given by

$$\boldsymbol{d}_E = e\tilde{F}_2(0)\boldsymbol{\sigma}\,. \tag{11.32}$$

The form factor $\tilde{F}_2(q^2)$ is therefore called the electric dipole form factor. Just like the anomalous magnetic moment, this can be non-zero even for an uncharged particle.

11.2.4 Anapole moment $\tilde{F}_3$

The matrix element for this term can be written as

$$\frac{e^{-iq\cdot x}}{\sqrt{2E_p V}\,\sqrt{2E_{p'}V}}\, e\tilde{F}_3(q^2)\left[\overline{u}_{s'}(\boldsymbol{p}')\gamma^\nu\gamma_5 u_s(\boldsymbol{p})\right]\ (q_\mu q_\nu - g_{\mu\nu}q^2). \quad (11.33)$$

Using the momentum-space version of the electromagnetic field equation given in Eq. (8.8), we find the contribution of this term to the $-j^\mu A_\mu$ term in the Lagrangian to be

$$-\frac{e^{-iq\cdot x}}{\sqrt{2E_p V}\,\sqrt{2E_{p'}V}}\, e\tilde{F}_3(q^2)\left[\overline{u}_{s'}(\boldsymbol{p}')\gamma^\nu\gamma_5 u_s(\boldsymbol{p})\right]\ j_\nu(q)\,. \quad (11.34)$$

In the non-relativistic limit, using the Dirac-Pauli representation of the spinors given in Eq. (11.20), this can be written as

$$\frac{e^{-iq\cdot x}}{V}\, e\tilde{F}_3(q^2)\left[\chi^\dagger_{s'}\boldsymbol{\sigma}\cdot\boldsymbol{j}\chi_s\right]\,. \quad (11.35)$$

As with the other form factors, we can take an inverse Fourier transform of the constant term in the form factor, and in the co-ordinate space it gives a total Lagrangian of the form

$$e\tilde{F}_3(0)\boldsymbol{\sigma}\cdot\boldsymbol{j}\,, \quad (11.36)$$

which is obtained after multiplying the Lagrangian (density) by the volume V. This is thus an interaction of the spin of the particle with the current density, something which was absent in classical physics. The quantity $\tilde{F}_3(0)$ is given a name in analogy with the multipole moments — it is called the *anapole moment.*

We have used concepts from classical physics in interpreting these form factors. It should be remembered however that classically, dipole moments can exist only for extended charge distributions. A point particle in classical physics can have only the total charge, which of course does not depend on the momentum transfer q. Every other form factor, as well as the momentum dependence of all of them, is a hallmark of quantum physics.

□ **Exercise 11.2** Verify that the electric dipole and the anapole moment interactions violate parity invariance.

□ **Exercise 11.3** Using the general form of the electromagnetic vertex of a complex scalar field defined in Ex. 11.1 (p 224), find the relation between $F(0)$ and the charge of the particle.

11.3 Anomalous magnetic moment of the electron

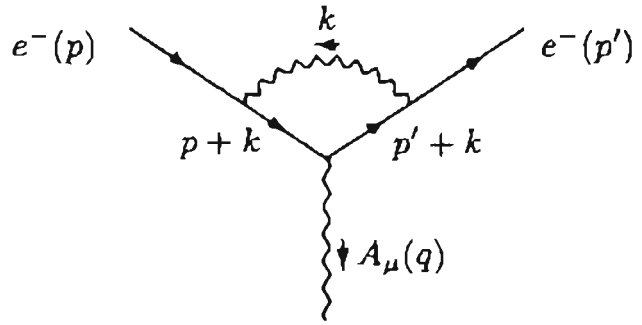


Figure 11.1: Lowest order correction to the vertex function in QED.

We now try to calculate the lowest order corrections to the vertex function of QED. At the 1-loop level, the contribution to the vertex comes from the Feynman diagram of Fig. 11.1. Notice that this diagram does not represent an S-matrix element, since the external photon line cannot be on-shell if the electron lines are on-shell, as discussed in §9.2. Nevertheless, we can calculate its amplitude using the Feynman rules of QED and obtain the effective term $-\langle j_\mu \rangle A^\mu$ in the Lagrangian. Omitting the external spinors from both sides, we can write the one loop contribution to Γ_μ as

$$-ie\Gamma_\mu^{(1)} = \int \frac{d^4k}{(2\pi)^4} ie\gamma_\lambda \, iS_F(p'+k) \, ie\gamma_\mu \, iS_F(p+k) ie\gamma_\rho \, iD^{\lambda\rho}(k) \,, \tag{11.37}$$

or

$$\Gamma_\mu^{(1)} = ie^2 \int \frac{d^4k}{(2\pi)^4} \, \frac{\gamma_\lambda(\not{p}' + \not{k} + m)\gamma_\mu(\not{p} + \not{k} + m)\gamma^\lambda}{[(p'+k)^2 - m^2]\,[(p+k)^2 - m^2]\,k^2} \,. \tag{11.38}$$

The task is now to evaluate this integral. We know from Ch. 10 that QED by itself does not violate parity, and from Ex. 11.2 (this page) that non-zero $\tilde{F}_2$ or $\tilde{F}_3$ indicate parity violation. So $\tilde{F}_2$ and $\tilde{F}_3$ can appear only if we include other interactions which violate parity. In pure QED, we have only the form factors F_1 and F_2. In the evaluation, we will concentrate on extracting the anomalous magnetic moment form factor F_2, the one associated with $i\sigma_{\mu\nu}q^\nu$. The evaluation of F_1 has some intricacies which will be pointed out in §11.4.

Feynman parameters for the denominator

This is the first time in this book that we are trying to evaluate the amplitude of a loop diagram. Therefore let us discuss some general techniques which will be useful for evaluation of any loop diagram. One of these is the introduction of Feynman parameters.

For this we need the identity

$$\frac{1}{a_1 a_2} = \int_0^1 d\zeta \, \frac{1}{[\zeta a_1 + (1-\zeta)a_2]^2}\,, \tag{11.39}$$

which can easily be checked by evaluating the integral on the right. Alternatively, by introducing a δ-function on the right hand side, we can write

$$\frac{1}{a_1 a_2} = \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \, \frac{\delta(1-\zeta_1-\zeta_2)}{[\zeta_1 a_1 + \zeta_2 a_2]^2}\,. \tag{11.40}$$

More generally, for any number of factors on the left, we can write

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \cdots \int_0^1 d\zeta_n \, \frac{\delta\left(1-\sum_{i=1}^n \zeta_i\right)}{\left[\sum_{i=1}^n \zeta_i a_i\right]^n}\,. \tag{11.41}$$

The parameters ζ_i are called *Feynman parameters*. We now demonstrate their use.

□ **Exercise 11.4** Prove Eq. (11.41).

For the integral in Eq. (11.38), we have three factors in the denominator, so we should use Eq. (11.41) with $n = 3$ to write

$$\frac{N_\mu(k)}{[(p'+k)^2 - m^2]\,[(p+k)^2 - m^2]\,k^2} = 2\int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \int_0^1 d\zeta_3 \, \delta(1-\zeta_1-\zeta_2-\zeta_3)\, \frac{N_\mu(k)}{D^3}\,, \tag{11.42}$$

where N_μ is the numerator in Eq. (11.38), and the denominator is given by

$$\begin{aligned} D &= \zeta_1[(p'+k)^2 - m^2] + \zeta_2[(p+k)^2 - m^2] + \zeta_3 k^2 \\ &= k^2 + 2k \cdot (\zeta_1 p' + \zeta_2 p)\,. \end{aligned} \tag{11.43}$$

In the second step, we have used that $\zeta_1 + \zeta_2 + \zeta_3 = 1$ because of the δ-function appearing in Eq. (11.42), and that $p^2 = p'^2 = m^2$.

At this stage, it seems that we have actually made our task more complicated. Instead of performing the integral that we had in Eq. (11.38), we have introduced three more integrations over the Feynman parameters. The advantage of this procedure will be obvious if we introduce a new momentum variable k' defined by

$$k' = k + \zeta_1 p' + \zeta_2 p\,. \tag{11.44}$$

In terms of this variable, we can write

$$D = k'^2 - (\zeta_1 p' + \zeta_2 p)^2\,, \tag{11.45}$$

so that the integral of Eq. (11.38) can be rewritten in the form

$$\begin{aligned} \Gamma_\mu^{(1)} = 2ie^2 \int \frac{d^4k'}{(2\pi)^4} \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \int_0^1 d\zeta_3 \\ \times \delta(1 - \zeta_1 - \zeta_2 - \zeta_3)\, \frac{N_\mu(k' - \zeta_1 p' - \zeta_2 p)}{[k'^2 - (\zeta_1 p' + \zeta_2 p)^2]^3}\,. \end{aligned} \tag{11.46}$$

This shows the first advantage of using the Feynman parameters. The denominator is now an even function of the new momentum variable k'. Thus terms in the numerator which are odd in k' will vanish on integration. Moreover, for the remaining terms, the integration over k' can be done very easily, as we will see shortly.

Let us first look at the numerator. Since k' is a dummy variable, the final result will not depend on it. So let us write it simply as k, because the old variable k will never be used again. Integration over the Feynman parameter ζ_3 gives us the step function, so we obtain

$$\begin{aligned} \Gamma_\mu^{(1)} = 8\pi i\alpha \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\zeta_1 \int_0^1 d\zeta_2\, \Theta(1 - \zeta_1 - \zeta_2) \\ \times \frac{N_\mu(k - \zeta_1 p' - \zeta_2 p)}{[k^2 - (\zeta_1 + \zeta_2)^2 m^2 + \zeta_1\zeta_2 q^2]^3}\,. \end{aligned} \tag{11.47}$$

Here we have used $p^2 = p'^2 = m^2$ in the denominator, which also implies that $q^2 = 2m^2 - 2p \cdot p'$, so that $2p \cdot p' = 2m^2 - q^2$. We have also used the fine-structure constant α in the prefactor, which, in our units, is given by $e^2/4\pi$.

The numerator

In the numerator of Eq. (11.42), we must replace k by $k - \zeta_1 p' - \zeta_2 p$, as indicated in Eq. (11.47). This gives

$$N_\mu(k - \zeta_1 p' - \zeta_2 p) = \gamma_\lambda \left[\not{a} + \not{k} + m\right] \gamma_\mu \left[\not{b} + \not{k} + m\right] \gamma^\lambda , \quad (11.48)$$

where for the sake of brevity we have written

$$\begin{aligned} a_\mu &= (1 - \zeta_1) p'_\mu - \zeta_2 p_\mu , \\ b_\mu &= (1 - \zeta_2) p_\mu - \zeta_1 p'_\mu . \end{aligned} \quad (11.49)$$

Thus we can write

$$\begin{aligned} N_\mu = {} & \gamma_\lambda \not{k} \gamma_\mu \not{k} \gamma^\lambda + \gamma_\lambda \not{a} \gamma_\mu \not{b} \gamma^\lambda \\ & + m \gamma_\lambda \left(\not{a} \gamma_\mu + \gamma_\mu \not{b}\right) \gamma^\lambda + m^2 \gamma_\lambda \gamma_\mu \gamma^\lambda , \end{aligned} \quad (11.50)$$

apart from terms linear in k, which integrate to zero when we perform the k integration, as mentioned earlier.

We now use the contraction relations of γ-matrices given in Eq. (4.42) to simplify the numerator further. Take for example the last term of Eq. (11.50). This can be rewritten as

$$-2m^2 \gamma_\mu , \quad (11.51)$$

by using the contraction formula with three γ-matrices. This term therefore does not contribute to the anomalous magnetic moment, and we will ignore this term for our present purpose. Similarly, the first term on the right side of Eq. (11.50), which contains two factors of the loop momentum, can be shown to contribute only to F_1. We will show this explicitly in §11.4 when we discuss the contribution to F_1. The other two terms can be simplified by the use of the contraction formulas in Eq. (4.42). This gives

$$N_\mu = -2 \not{b} \gamma_\mu \not{a} + 4m \left(a_\mu + b_\mu\right) + \cdots , \quad (11.52)$$

where the dots, for the rest of this section, stand for any term which we know do not contain any contribution to the anomalous magnetic moment.

To extract the terms contributing to the form factor F_2 from these remaining terms in the numerator, we note two things. First, ζ_1 and ζ_2 can be interchanged since they are dummy variables. Also, it is only the numerator which changes under this exchange. Thus, for example, the second term in Eq. (11.52) can be written as

$$4m\left[(1-2\zeta_1)p'_\mu + (1-2\zeta_2)p_\mu\right] = 4m(1-\zeta_1-\zeta_2)(p+p')_\mu\,. \quad (11.53)$$

Remembering that this appears in the expression for Γ_μ, which is sandwiched between the spinors, we can use the Gordon identity to write this term as

$$4m(1-\zeta_1-\zeta_2)\left[2m\gamma_\mu + i\sigma_{\mu\nu}q^\nu\right]\,. \quad (11.54)$$

Only the second term in this contributes to F_2.

The second thing to note is that, since the whole expression has $\overline{u}(p')$ to the left and $u(p)$ to the right, we can use the definitions of the spinors to write $\overline{u}(p')\not{p}' = m\overline{u}(p')$ and $\not{p}u(p) = mu(p)$. In other words, if we obtain a factor of $\not{p}'$ at the extreme left of the expression for Γ_μ, we can replace it by the mass m. Similarly, if we obtain a factor of $\not{p}$ at the extreme right, we can replace it by m as well. With this observation, we see that in the first term on the right side of Eq. (11.52), we can write

$$\begin{aligned}\not{p} &= (1-\zeta_2)\not{p} - \zeta_1\not{p}' \\ &= (1-\zeta_1-\zeta_2)\not{p}' + (1-\zeta_2)\not{q} \\ &= (1-\zeta_1-\zeta_2)m + (1-\zeta_2)\not{q}\,. \end{aligned} \quad (11.55)$$

As we already said, the last step does not follow in general, but is valid when the entire expression has the $\overline{u}(p')$ to the left. Similarly, we can write

$$\not{p} = -(1-\zeta_1)\not{q} + (1-\zeta_1-\zeta_2)m\,. \quad (11.56)$$

In the combination $\not{p}\gamma_\mu\not{p}$ which appears in Eq. (11.52), we now obtain four terms. Among these, the term containing both factors of m from the expressions for $\not{p}$ and $\not{p}$ contains only a γ_μ, and therefore

contributes only to the form factor F_1. Then there is another term which contains $\not{q}\gamma_\mu\not{q}$, which also does not contribute to F_2. We are thus left with the terms with one factor of m and one factor of $\not{q}$. Again interchanging ζ_1 and ζ_2 in one of these terms, we can write

$$\begin{aligned} -2\not{p}\gamma_\mu\not{q} &= 2m(1-\zeta_1-\zeta_2)(1-\zeta_1)\,(\gamma_\mu\not{q}-\not{q}\gamma_\mu)+\cdots \\ &= -4m(1-\zeta_1-\zeta_2)(1-\zeta_1)i\sigma_{\mu\nu}q^\nu+\cdots\,. \end{aligned} \tag{11.57}$$

Adding the two contributions from Eqs. (11.54) and (11.57), we can extract the general expression for $F_2(q^2)$. For the anomalous magnetic moment, we need just $F_2(0)$, which is

$$F_2(0) = 32\pi i m\alpha \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\zeta_1 \int_0^{1-\zeta_1} d\zeta_2 \, \frac{\zeta_1(1-\zeta_1-\zeta_2)}{[k^2-(\zeta_1+\zeta_2)^2m^2]^3}\,, \tag{11.58}$$

where we have used the step function to adjust the limits of the ζ_2 integration.

Wick rotation and momentum integration

We now want to perform the integrations. As we mentioned earlier, the whole point of using the Feynman parameters is that the momentum integration can be done first, and done easily. So let us try that.

Let us first recall that the denominator in the expression for F_2 came from the denominator of various propagators. While writing the propagators, we consistently ignored the $i\varepsilon$ term. Partly this was done to save writing, knowing well that we could bring it back whenever necessary. The need arises now because we have to integrate over all values of the loop momentum. Looking at Eq. (11.38), we notice that since p and p' are on-shell momenta, the internal fermion propagators would diverge near $k=0$. With the redefined loop momentum, we have an expression of the form

$$I_s(A) \equiv \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{[k^2-A^2+i\varepsilon]^s}\,, \tag{11.59}$$

for a suitably defined quantity A^2. The power s equals 3 for the problem at hand, but our argument will hold for other positive integers, as discussed in §A.4 of the Appendix. The integrand has poles

for certain values of k^2. Since A^2 is real, if we did not have the little imaginary part in the denominator, these poles would have been on the real axis, and would pose problems while we tried to perform the integration. This predicament was discussed while we found out the propagators for various fields in earlier chapters of this book.

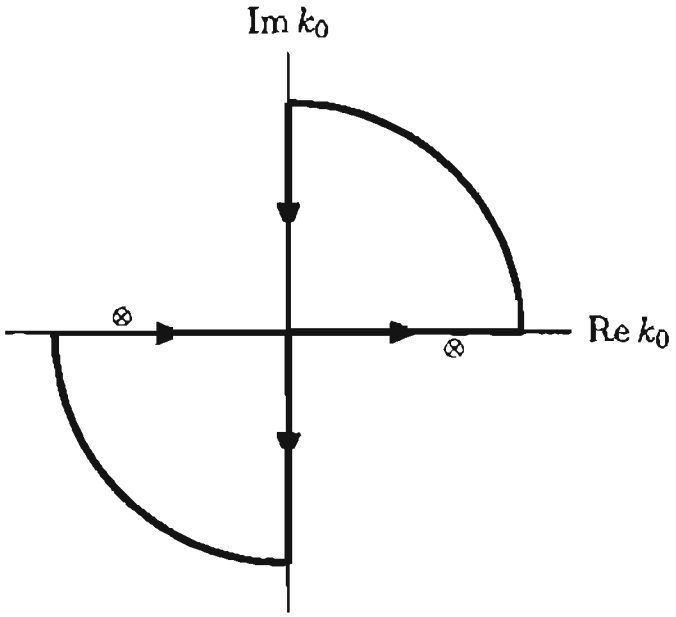


Figure 11.2: The contour for performing the k_0 integration is shown with thick lines. The crosses indicate the poles of the integrand.

Once we have the imaginary part, the poles are not on the real axis and we have no problem. Suppose first we are trying to integrate over k_0. The denominator in this case can be written as $k_0^2 - B^2 + i\varepsilon$, where $B^2 = \boldsymbol{k}^2 + A^2$. If we consider complex values of k_0, the poles are at $B - i\varepsilon$ and $-B + i\varepsilon$, as shown by the little crosses in Fig. 11.2.

We have been careless about the factors going with the ε term in the denominator. This is because ε is a very small parameter, which should be taken to zero at the end of the calculation anyway. The only thing to remember is that it has to be taken to zero from the positive side. Thus, we have to be careful that the quantity which may multiply ε is positive, and we have been careful about that.

In any case, let us turn to the integration over k_0. Imagine that we are performing it for the integrand which appears in Eq. (11.59), but over the contour in the complex k_0-plane. The contour we have

in mind is shown in Fig. 11.2. Since this is a closed contour and it does not contain any of the poles of the integrand, the integral should vanish, irrespective of the radius of the curved regions of the contour.

Consider this statement when this radius is infinitely large. In that case, the contribution to the integral from the curved regions is infinitesimally small. Thus, the vanishing of the integral would mean that the contributions from the two straight lines add up to zero, i.e.,

$$\int_{-\infty}^{+\infty} dk_0 \frac{1}{(k_0^2 - B^2 + i\varepsilon)^s} + \int_{+i\infty}^{-i\infty} dk_0 \frac{1}{(k_0^2 - B^2 + i\varepsilon)^s} = 0\,. \quad (11.60)$$

In the second term, we can change the integration variable from k_0 to a new variable k_{0E}, defined by $k_0 = ik_{0E}$. This allows us to write

$$\int_{-\infty}^{+\infty} dk_0 \frac{1}{(k_0^2 - B^2 + i\varepsilon)^s} = i \int_{-\infty}^{+\infty} dk_{0E} \frac{1}{(-k_{0E}^2 - B^2 + i\varepsilon)^s}\,. \quad (11.61)$$

Reinstating the integrations over the spatial components of k, we can then write Eq. (11.59) as

$$I_s(A) = i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{\left[-k_E^2 - A^2 + i\varepsilon\right]^s}\,, \quad (11.62)$$

where now

$$k_E^2 = k_{0E}^2 + \boldsymbol{k}^2\,. \quad (11.63)$$

The advantage of this form over the earlier one is that k_E^2 is always non-negative, so we can apply techniques which are familiar to us from our experience of spatial integration. This method of using the complex analysis to bring in k_E^2 in place of k^2 is called *Wick rotation*. Of course, it does not refer to any physical rotation. It means that, in the complex plane, we have chosen a new path of integration which is rotated with respect to the original one along the real axis.

The method for performing this integral is described in detail in Appendix A.4.2. Using Eq. (A.54) here, we can write down the result:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 - A^2 + i\varepsilon\right]^3} = -\frac{i}{32\pi^2} \frac{1}{A^2}\,. \quad (11.64)$$

Integration over Feynman parameters

We now put the result of the momentum integration into Eq. (11.58). Since $A^2 = (\zeta_1 + \zeta_2)^2 m^2$ in our case, we obtain

$$F_2(0) = \frac{\alpha}{\pi m} \int_0^1 d\zeta_1 \int_0^{1-\zeta_1} d\zeta_2 \, \frac{\zeta_1(1-\zeta_1-\zeta_2)}{(\zeta_1+\zeta_2)^2} \,. \tag{11.65}$$

The integration is straight forward, and yields the result

$$F_2(0) = \frac{\alpha}{4\pi m} \,. \tag{11.66}$$

Plugging this back into Eq. (11.30) and remembering that the charge of the electron is $-e$, i.e., $Q = -1$ for the electron, we obtain the Landé g-factor to be

$$g = 2 + \frac{\alpha}{\pi} \,. \tag{11.67}$$

The second term is the anomalous contribution of the g-factor, which was first calculated by Schwinger. Of course there are further corrections to the g-factor, which comes from higher order diagrams, and they contain higher powers of α.

Calculation of the anomalous magnetic moment of the electron has been one of the greatest triumphs of QED. To date, the calculations have been done up to the order α^4. The results give the theoretical prediction to be

$$\frac{1}{2}(g-2) = 1\,159\,652\,460(127)(75) \times 10^{-12} \,. \tag{11.68}$$

The uncertainties for the last digits are shown in parentheses — the first one due to the experimental uncertainty in α and the second one due to computational limitations. The experimentally measured value for this quantity is

$$\frac{1}{2}(g-2) = 1\,159\,652\,193(10) \times 10^{-12} \,. \tag{11.69}$$

Thus we see that the two agree up to seven significant digits! The anomalous moment of the muon agrees with experiment up to eight significant digits.

11.4 Charge form factor

Since QED is parity invariant, the only other form factor that appears in pure QED interactions is the charge form factor F_1. Let us now consider the contributions to F_1. These are terms in Eq. (11.50) which are proportional to γ_μ. The last term reduces to the expression of Eq. (11.51) and so contributes only to F_1. The contribution of the third term can be read from Eq. (11.54), and it is $8m^2(1-\zeta_1-\zeta_2)\gamma_\mu$. As for the second term, i.e., the term $-2\not{p}\gamma_\mu\not{p}$, we have indicated which terms do not contribute to F_2. These can now be assembled:

$$-2\not{p}\gamma_\mu\not{p} = -2m^2(1-\zeta_1-\zeta_2)^2\gamma_\mu + 2(1-\zeta_1)(1-\zeta_2)\not{q}\gamma_\mu\not{q} + \cdots , \tag{11.70}$$

where in this section, the dots mean the terms which are irrelevant for F_1. Using the anticommutation relation of the γ-matrices, we can write $\not{q}\gamma_\mu\not{q} = (2q_\mu - \gamma_\mu\not{q})\not{q} = 2q_\mu\not{q} - \gamma_\mu q^2$. Of these, the first term vanishes between the two spinors, and the second term contributes to F_1. Thus we can write

$$-2\not{p}\gamma_\mu\not{p} = -2m^2(1-\zeta_1-\zeta_2)^2\gamma_\mu - 2q^2(1-\zeta_1)(1-\zeta_2)\gamma_\mu + \cdots . \tag{11.71}$$

We now have to discuss the first term in Eq. (11.50). This can be written as

$$k^\nu k^\rho \gamma_\lambda\gamma_\nu\gamma_\mu\gamma_\rho\gamma^\lambda = -2k^\nu k^\rho\gamma_\rho\gamma_\mu\gamma_\nu . \tag{11.72}$$

using the contraction formula from Eq. (4.42). Thus for this term the integration over k has the following generic form:

$$I^{\nu\rho} = \int d^4k\, k^\nu k^\rho f(k^2) , \tag{11.73}$$

where $f(k^2)$ is some function of k^2. The integral must be symmetric in the indices ν and ρ and obviously cannot depend on the 4-vector k which has been integrated over. Thus it must be proportional to the metric tensor $g^{\nu\rho}$, and we can write

$$I^{\nu\rho} = g^{\nu\rho} J . \tag{11.74}$$

Contracting both sides of this equation with $g_{\nu\rho}$ and using the fact that $g_{\nu\rho}g^{\nu\rho} = 4$, we obtain

$$J = \frac{1}{4} g_{\nu\rho} I^{\nu\rho} = \frac{1}{4} \int d^4k \; k^2 f(k^2) \,, \tag{11.75}$$

using the definition of $I^{\nu\rho}$ from Eq. (11.73). Thus we can write

$$\int d^4k \; k^\nu k^\rho f(k^2) = \frac{1}{4} g^{\nu\rho} \int d^4k \; k^2 f(k^2) \,, \tag{11.76}$$

a result which is extremely useful for performing loop integrations.

□ **Exercise 11.5** Verify Eq. (11.76) explicitly by changing integration variables suitably. For this, first check that the integral on the left should vanish if $\nu \neq \rho$. Then, by change of variables, show that

$$\int d^4k \; (k^0)^2 f(k^2) = - \int d^4k \; (k^i)^2 f(k^2) \tag{11.77}$$

for any $i = 1, 2, 3$. Finally, add these four equal objects and divide by 4 to obtain Eq. (11.76).

□ **Exercise 11.6** Show that

$$\int d^4k \; k^\mu k^\nu k^\lambda k^\rho f(k^2) = \frac{1}{24} \left(g^{\mu\nu} g^{\lambda\rho} + g^{\mu\lambda} g^{\nu\rho} + g^{\mu\rho} g^{\nu\lambda} \right) \times \int d^4k \; k^4 f(k^2) \,. \tag{11.78}$$

Going back to Eq. (11.72), we now see that in the integrand, we can write the numerator for this term as

$$-\frac{1}{2} g^{\nu\rho} k^2 \gamma_\rho \gamma_\mu \gamma_\nu = -\frac{1}{2} k^2 \gamma_\rho \gamma_\mu \gamma^\rho = k^2 \gamma_\mu \,, \tag{11.79}$$

using the contraction formula from Eq. (4.42). Thus this term also contributes only to the charge form factor, as was claimed in §11.3.

Collecting all the terms, we now find that the contribution to $F_1(q^2)$ coming from the diagram of Fig. 11.1 is:

$$F_1(q^2) = 8\pi i\alpha \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \; \Theta(1 - \zeta_1 - \zeta_2) \times \frac{N_1}{\left[k^2 - (\zeta_1 + \zeta_2)^2 m^2 + \zeta_1 \zeta_2 q^2 \right]^3} \,, \tag{11.80}$$

where

$$N_1 = k^2 - 2q^2(1-\zeta_1)(1-\zeta_2) - 2m^2\left[(\zeta_1+\zeta_2)^2 - 2(1-\zeta_1-\zeta_2)\right]. \tag{11.81}$$

Notice that there is a term in N_1 which is just k^2. Consider the momentum integration for this term. Apart from an overall factor and the integrations over the Feynman parameters, we obtain an expression of the form

$$\int d^4k \, \frac{k^2}{\left[k^2 - (\zeta_1+\zeta_2)^2 m^2 + \zeta_1\zeta_2 q^2\right]^3}. \tag{11.82}$$

This is a function of q^2. Consider now the contribution to this integral coming from large values of k^2, i.e., for $|k^2| > M^2$, where M is a mass scale much larger than both m and $\sqrt{|q^2|}$. For this part, we can neglect m^2 and q^2 and write this part as

$$\int_{|k^2|>M^2} \frac{d^4k}{k^4}. \tag{11.83}$$

The indefinite integral will go like the logarithm of $|k^2|$. But the upper limit of this integration is, of course, infinite. Thus, once we perform the integration, we will obtain an infinite result. Such infinities are called *ultra-violet divergences* since they are caused by the large-k behavior of the integrand. Ultra-violet (UV) divergences appear in other diagrams as well. A careful and extended treatment of such divergences will be presented in Ch. 12.

11.5 Electron-proton scattering

Form factors are also useful in considering tree-level processes, when there are vertices involving fermions which are not elementary particles. As an example, we can consider the elastic scattering of electrons and protons. Protons are known to be composite objects, composed of quarks. Thus, protons are not simple Dirac particles. The only way to write down the electromagnetic vertex involving them is to use form factors.

To be more specific, let us look at Fig. 11.3, which gives the lowest order diagram for the scattering. The Feynman amplitude

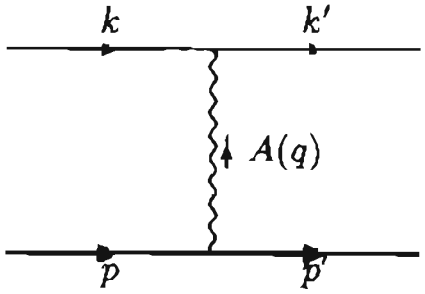


Figure 11.3: Lowest order diagram for electron-proton elastic scattering. The lighter lines denote electrons, the heavier lines protons. The momentum for each line is shown.

can be written as

$$i\mathscr{M} = \left[\overline{u}_{(e)}(k')\, ie\gamma^\mu\, u_{(e)}(k)\right] \frac{-ig_{\mu\nu}}{q^2} \left[\overline{u}_{(p)}(p')\,(-ie\Gamma^\nu)\, u_{(p)}(p)\right], \tag{11.84}$$

where the subscripts on the spinors denote the particle concerned. Here Γ^ν stands for the vertex function of the proton. Had the proton been a Dirac particle, we would have written $\Gamma^\nu = \gamma^\nu$ just as we have done for the electron. But as we said, the proton is not a Dirac particle. So we will have to use the general expression for the vertex function. Assuming parity invariance, the vertex function contains two form factors, F_1 and F_2:

$$\Gamma_\mu = \gamma_\mu F_1(q^2) + i\sigma_{\mu\lambda}q^\lambda F_2(q^2)\,. \tag{11.85}$$

As explained in §11.2, $F_1(0) = 1$ which gives the proton charge in units of e, and $F_2(0)$ is related to the anomalous magnetic moment of the proton. Indeed, proton has a large anomalous magnetic moment, indicated by the fact that its Landé g-factor is given by

$$g = 5.58\,, \tag{11.86}$$

whereas the Dirac contribution is only 2.

So we can write the Feynman amplitude as

$$\mathscr{M} = -\frac{e^2}{q^2}\left[\overline{u}_{(e)}(k')\,\gamma^\mu\, u_{(e)}(k)\right]\left[\overline{u}_{(p)}(p')\,\Gamma_\mu\, u_{(p)}(p)\right], \tag{11.87}$$

where Γ_μ is given by Eq. (11.85). For the sake of simplicity, let us neglect the electron mass and denote the proton mass by m_p. Then, averaging over initial spins and summing over final ones, we obtain

$$\overline{|\mathscr{M}|^2} = \frac{e^4}{4q^4} \ell^{\mu\nu} W_{\mu\nu} , \tag{11.88}$$

where

$$\begin{aligned} \ell^{\mu\nu} &= \mathrm{Tr}\left[\not{k}' \gamma^\mu \not{k} \gamma^\nu\right] \\ &= 4\left[k'^\mu k^\nu + k^\mu k'^\nu + \frac{1}{2} q^2 g^{\mu\nu}\right] , \end{aligned} \tag{11.89}$$

and

$$W_{\mu\nu} = \mathrm{Tr}\left[(\not{p}' + m_p)\Gamma_\mu (\not{p} + m_p)\Gamma_\nu^\dagger\right] . \tag{11.90}$$

Using the form for Γ_μ given in Eq. (11.85), one can take the trace. The final result is

$$W_{\mu\nu} = 4\left[W_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2}{m_p^2}\left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right)\right], \tag{11.91}$$

where

$$\begin{aligned} W_1 &= -\frac{1}{2} q^2 (F_1 + 2m_p F_2)^2 , \\ W_2 &= 2m_p^2 (F_1^2 - q^2 F_2^2) . \end{aligned} \tag{11.92}$$

The calculation of the differential cross section is now straightforward. In the Lab frame where the initial proton is at rest, it comes out to be

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8m_p^2 E^2} \frac{2W_1 \sin^2\frac{\theta}{2} + W_2 \cos^2\frac{\theta}{2}}{\left[1 + (2E/m_p)\sin^2\frac{\theta}{2}\right]\sin^4\frac{\theta}{2}} , \tag{11.93}$$

where E is the energy of the incident electron.

There is a big advantage of writing the differential cross section in terms of W_1 and W_2 rather than F_1 and F_2. Suppose we now consider the inelastic process

$$e^- + p \to e^- + X , \tag{11.94}$$

where X represents any number of particles. In this case, we cannot characterize the vertex in terms of a vertex function of the form of Eq. (11.85). However, even in this case, Lorentz invariance implies that the square of the Feynman amplitude must be expressible in the form given in Eq. (11.88), with $\ell^{\mu\nu}$ given by Eq. (11.89). Moreover, due to current conservation, $W_{\mu\nu}$ must satisfy the conditions

$$q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0\,. \tag{11.95}$$

In addition, $W_{\mu\nu}$ can depend only on the vectors k and p, or equivalently on q and p. The most general tensor constructed from two vectors and satisfying Eq. (11.95) has the form

$$W_{\mu\nu} = 4\left[W_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2}{m_p^2}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) + \frac{W_3}{2m_p^2}\varepsilon_{\mu\nu\lambda\rho}p^\lambda q^\rho\right]. \tag{11.96}$$

If parity invariance holds, $W_3 = 0$, so we are left with the form of Eq. (11.91). The differential cross section is still of the form of Eq. (11.93), although W_1 and W_2 cannot be expressed as in Eq. (11.92).

□ **Exercise 11.7** The pions are spin-0 particle which, like protons, are composite objects. Use the parametrization of the vertex given in Ex. 11.1 (p 224) to calculate the differential cross section of elastic scattering of electrons off charged pions. Show that

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2}\,\frac{F^2\cos^2\dfrac{\theta}{2}}{\left[1 + (2E/m_\pi)\sin^2\dfrac{\theta}{2}\right]\sin^4\dfrac{\theta}{2}}\,. \tag{11.97}$$

Chapter 12

Renormalization

In calculating various amplitudes, we have encountered two kinds of divergences. In §9.10, we found an infra-red divergence, i.e., an amplitude which becomes infinite for vanishingly small values of some momentum. On the other hand, in §11.4 we saw an example of ultra-violet divergence, which is caused by arbitrarily large values of momenta in a loop integration. In this chapter, we take up the issue of how these infinities can be interpreted. As a paradigm, we will often use QED as the interacting theory, but some of the general arguments will be valid for any field theory.

Our primary concern will be the elimination of the ultra-violet divergences. In fact, unless otherwise specified, "divergence" will mean ultra-violet divergence in this chapter. We will see that the structure needed to interpret ultra-violet divergences will also allow the cancellation of infra-red divergences.

12.1 Degree of divergence of a diagram

12.1.1 Superficial degree of divergence

Let us find out which diagrams are divergent, i.e., make infinite contributions to the amplitude. As we already saw in §11.3, the infinities in an amplitude come from momentum integration.

So let us consider a generic diagram with ℓ loops, in which the number of internal fermion lines is n_f and the number of internal boson lines is n_b. Let us also say that in this generic diagram there are different kinds of vertices and the i-th kind appears v_i times.

In a generic field theory, the interaction terms in the Lagrangian which give rise to these vertices may also involve derivatives. Let the number of derivatives in the i-th type of vertex be d_i.

Now suppose we have written down the Feynman amplitude of this diagram and we are trying to see how it behaves when the internal momenta are in the ultra-violet region, i.e., are much larger than any of the masses of the particles in the loops. Each internal fermion line contributes a propagator which behaves like $1/p$. For internal scalar lines, the propagators behave like $1/p^2$ at large momentum. Let us assume that the internal vector boson propagators also behave the same way, as was the case for the photon propagator given in Eq. (8.40). Integration over each momentum loop has four powers of momentum due to the d^4p, and each derivative in the interaction terms contribute one power of p. We can therefore count the overall power of momentum in the integral,

$$D = 4\ell - 2n_b - n_f + \sum_i v_i d_i \,. \tag{12.1}$$

This quantity D is called the *superficial degree of divergence* of the diagram. If this turns out to be negative, the corresponding amplitude is expected to be convergent. If on the other hand D is positive, the amplitude apparently diverges when we consider the integration up to infinitely large values of the loop momenta. Even if D is zero, the integral can be logarithmically divergent. This was the case of the diagram for the electron-photon vertex in Fig. 11.1, which had only one kind of vertex with $d = 0$, and the diagram had $\ell = 1$, $n_b = 1$, $n_f = 2$, so that $D = 0$. Indeed, we observed in §11.4 that the form factor F_1 in this diagram turned out to be logarithmically divergent.

The expression for the superficial degree of divergence given in Eq. (12.1) can be written in a more convenient form if we try to eliminate the number of internal lines from the expression and use the number of external lines instead. For this, we use the relation mentioned earlier in Eq. (6.56):

$$\sum_i v_i - n_b - n_f - 1 = -\ell \,, \tag{12.2}$$

with a trivial change of notation. Let us also suppose that there are f_i fermion lines and b_i boson lines at a vertex of the i-th type. If

there are E_b external boson lines and E_f external fermion lines, we must have the relations

$$E_b + 2n_b = \sum_i v_i b_i \,,$$
$$E_f + 2n_f = \sum_i v_i f_i \,, \tag{12.3}$$

because each internal line must connect two vertices and each external line attaches to only one vertex. Using these we can eliminate ℓ, n_b and n_f from Eq. (12.1) to get

$$D = 4 - E_b - \frac{3}{2}E_f + \sum_i v_i \left(d_i + b_i + \frac{3}{2} f_i - 4 \right) . \tag{12.4}$$

The quantity multiplying v_i in the last term has a simple interpretation. The mass dimension of the field operators, including the derivatives, at any vertex of the i-th type is $d_i + b_i + \frac{3}{2} f_i$, and thus the coupling constant associated with this interaction must have mass dimension

$$\delta_i = 4 - d_i - b_i - \frac{3}{2} f_i \,. \tag{12.5}$$

Putting everything together we can write the superficial degree of divergence in the form

$$D = 4 - E_b - \frac{3}{2}E_f - \sum_i v_i \delta_i \,. \tag{12.6}$$

Let us now see what this implies. Suppose we have a theory in which all the interactions have $\delta_i \geq 0$. Consider a diagram involving E_b external bosonic lines and E_f external fermionic lines. If E_b, or E_f, or both, are large enough, the degree of divergence becomes negative and the Feynman amplitude for this diagram is superficially convergent. The degree of divergence can be positive or zero at most for a finite number of choices of E_b and E_f. This means that the theory has only a finite number of divergent amplitudes. If we can figure out how to make sense of these few divergent amplitudes, we can expect to get meaningful physical quantities from a theory in which $\delta_i \geq 0$. There is a consistent procedure for doing this, called *renormalization*, which we will describe in the rest of the chapter.

Theories in which the coupling constants have zero or positive mass dimension are called power-counting renormalizable.

On the other hand, consider a theory where there is at least one interaction for which $\delta_i < 0$. In this case, no matter what values of E_b and E_f we consider, we can always find a diagram, with large enough value of v_i, which will make the degree of divergence non-negative. In other words, there will be an infinite number of infinite amplitudes. And since the superficial degree of divergence is different for diagrams with different number of external lines, the divergent integrals in these amplitudes are all different. In such a case, there is no consistent way to deal with the divergences, and the theory is called *non-renormalizable*. Such theories would either be unacceptable physical theories, or may be valid as an approximate theory only in a specified momentum range, so that the momentum integrations do not go to large values and the problem of infinities does not arise.

□ **Exercise 12.1** Consider a theory with a massive vector boson whose propagator is $\mathcal{O}(1)$ for large momenta, as shown in Eq. (8.22). Show that the superficial degree of divergence for this theory is given by

$$D = 4 - E_\phi - \frac{3}{2}E_f - 2E_A + \sum_i v_i(a_i - \delta_i)\,, \tag{12.7}$$

where E_ϕ and E_A are the number of external scalar and vector lines, a_i is the number of vector boson lines at a vertex of the i-th type, and all other symbols have the meaning given in the text. From this, argue that the theory would not be power-counting renormalizable.

12.1.2 Superficial vs. real degree of divergence

In the entire discussion above, we have used the word "superficial" several times. Consider a diagram for which the value of D, calculated from Eq. (12.6), is positive or zero. Does it necessarily imply that the amplitude of this diagram is divergent? Not really, because in finding D, we have counted the powers of momentum in the integrand and in the measure of the integral. If all of these momenta are internal, then of course the amplitude would be infinite. But we do not know if that is the case. If for some reason the amplitude has to contain one or more powers of external momenta, the number of powers of internal momenta may be negative, and the amplitude would be finite. We will see such a case in QED later, where the amplitude

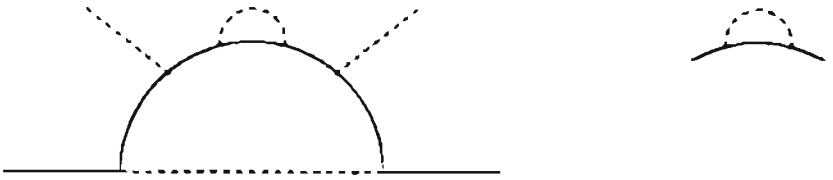


Figure 12.1: An example of a superficially convergent diagram with a divergent subdiagram.

is finite despite the fact that the superficial degree of divergence is zero.

There are other ways in which a superficially divergent diagram can be convergent. Consider, for example, supplementing the QED Lagrangian with a photon mass term $\frac{1}{2}M^2 A_\mu A^\mu$. It would appear from Eq. (12.7) that this theory is non-renormalizable. However, the part of the propagator which behaves as $\mathcal{O}(1)$ at large momenta is proportional to $k_\mu k_\nu$. Since the photon always couples to conserved currents, this part does not contribute to any amplitude. So effectively the propagator still behaves as $1/k^2$ in the ultra-violet region, and the theory is power-counting renormalizable.

There is a flip side of "superficiality". Consider a diagram now for which D is negative, so that the diagram is superficially finite. But the quantity D counts the overall powers of momentum. In a diagram with more than one loops, there are more than one internal 4-momenta which have to be integrated. Even if the total degree of divergence is negative, it is certainly possible that if we consider only the powers of one particular internal 4-momentum, the net power of this momentum in the integral is zero or positive. In that case, integration over that particular 4-momentum will diverge. For example, consider the 2-loop diagram of Fig. 12.1a in the Yukawa theory of scalars and fermions discussed in §6.1. The diagram has $D = -1$ so that it is superficially convergent. However, the loop shown separately in Fig. 12.1b is a subdiagram with $D = +1$, and the momentum integration over it diverges. Clearly, a diagram containing a divergent subdiagram can be superficially convergent even if the number of loops is the same for both. For example, the 1-loop diagrams in Fig. 6.5 have $D = -2$ and are superficially convergent. But the loop itself, shown in Fig. 6.6, is a subdiagram with $D = 0$

and is therefore divergent.

However, we can ignore such hidden divergences when we try to apply the renormalization procedure order by order in perturbation theory. The reason is simple. If there is a divergent subdiagram in a superficially convergent diagram, that subdiagram will be simpler than the overall diagram in the sense that it will contain either smaller number of loops, or smaller number of external legs, or both. If at the level of discussing these simpler diagrams, we have already decided how to interpret the infinities occurring there, we can just use that wisdom in this larger diagram and obtain a result that is physically meaningful.

12.2 Specific examples in QED

We now discuss the implications of our discussion on QED, which is the theory with a spinor and the photon field, with the interaction term given by

$$\mathscr{L}_{\text{int}} = -eQ\overline{\psi}\gamma_\mu\psi A^\mu \tag{12.8}$$

The coupling constant associated with this interaction is dimensionless. Therefore, $\delta = 0$. The formula for the superficial degree of divergence is given simply by

$$D = 4 - E_b - \frac{3}{2}E_f\,, \tag{12.9}$$

where E_b is now the number of external photon lines and E_f, as before, is the number of external fermion lines. Because of charge conservation, E_f must be even. In addition, charge conjugation invariance of QED implies that any amplitude with $E_f = 0$ and odd E_b has to vanish. Therefore we can immediately conclude that only the following amplitudes have non-negative superficial degrees of divergence:

E_b	E_f	D
0	2	1
2	0	2
1	2	0
4	0	0

(12.10)

Of these, the last one is not really divergent. The reason is the gauge invariance of QED, which demands that if *all* external lines of a diagram are photon lines, the amplitude should vanish when contracted with the momentum of any of those lines. We will illustrate later that this implies that the amplitude of a diagram with n external photon lines must contain at least n powers of external momenta. Thus for the amplitude with four external photons, the real degree of divergence cannot be larger than -4. Hence this amplitude must be convergent. For the amplitude with two external photon lines, we can also say that there must be at least two external momenta, but this reduces the degree of divergence from 2 to zero, and therefore the amplitude can still be divergent. Thus we have found that there are three kinds of amplitude which can be divergent in QED, viz., those corresponding to the first three rows of the table in Eq. (12.10).

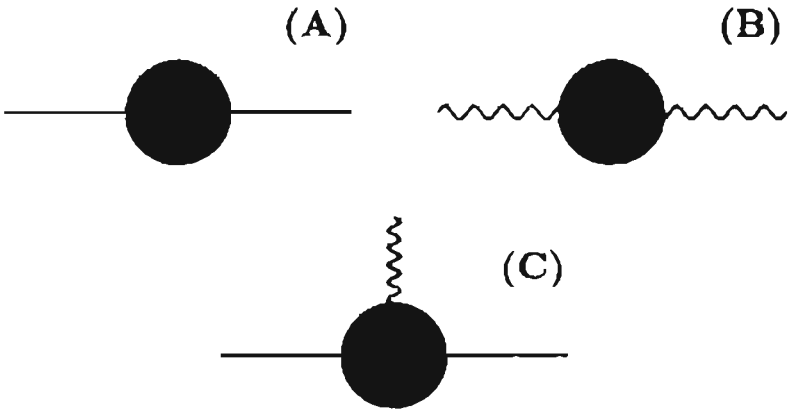


Figure 12.2: Schematic representations of divergent amplitudes in QED. The blobs stand for any number of internal lines.

In Fig. 12.2, we show these amplitudes in a schematic way. The diagram marked (A) with two external fermion lines is called the *self-energy diagram* for the fermion. By the same token, diagram (B) should be called the *self-energy diagram* for the photon. This name is used sometimes, but the name *vacuum polarization diagram* is more popular.

The last diagram, which has two fermion lines and a photon line, is called the *vertex diagram* because it contributes to the interaction vertex of QED. In Ch. 11 we have discussed the 1-loop contribution to such amplitudes and found that the amplitude indeed turns out to be infinite. This infinity showed up in the form factor F_1, which

we discussed in §11.4. In the rest of this chapter, we will try to figure out a way to consistently interpret the infinite amplitudes.

12.3 Outline of the program

The infinite amplitudes are tackled in two steps. In the first step, the goal is to somehow perform the loop integrations and write it in a closed form by introducing some extra parameter in the theory. The parameter is introduced in a way that the integrals can be performed for a range of its values. However, if we were to take the values appropriate for the physical limit, the result would be infinite. This part of the process is called *regularization*.

There are many ways in which regularization can be performed. For example, we can choose to integrate all loop momenta up to a certain maximum value of Λ. Then we can perform the integrations and would obtain results which depend on Λ. This procedure is called a *cut-off regularization*. Alternatively, we can use some auxiliary fields which do not actually appear in the physical theory. These are introduced in such a way that the infinities occurring from physical particles in the loop are canceled by the infinities arising from these auxiliary particles in the loop. The integrations can then be performed and the results depend on the masses of these auxiliary particles. The infinities reappear in the limit when the masses of these auxiliary particles are set to infinity. This procedure is called *Pauli-Villars regularization*. A third procedure is to formally make the number of space-time dimensions an arbitrary real number. The results of integrations now would depend on this number. When it is set equal to four, the infinities show up. This process is called the *dimensional regularization*. In all cases, the parameter dependent part which becomes infinite in the physical limit is called the divergent part of the amplitude.

Once we have regularized a divergent integral to a parameter dependent finite number, we can introduce what are called *counterterms* into the Lagrangian. These are products of the fields, with coefficients chosen such that the divergent part of each regularized amplitude is canceled by the contribution from the corresponding counterterm. This procedure is called *renormalization*. If all the counterterms are of the same form as some or all of the terms present

in the original Lagrangian, this is equivalent to only a rescaling of masses, coupling constants and fields by (parameter dependent, divergent) constant numbers. Theories for which this can be done are called *renormalizable* theories.

We will not discuss the cut-off regularization procedure in any detail because it is not explicitly Lorentz invariant. But we will use the other two types of regularization. And we will also show that the physical results are independent of the regularization procedure.

12.4 Ward-Takahashi identity

We already identified three types of divergent amplitudes for QED. There is in fact a relation between them which makes it easier to tackle them. This can be guessed from the fact that the interaction between fermions and photons was introduced by the minimal substitution method summarized in Eq. (9.15), which means $\partial\!\!\!/ \to \partial\!\!\!/ + ieQA\!\!\!/$, using the gauge principle.

The relation can be explicitly stated now. The minimal substitution method implies that the vertex function introduced in §11.1 is given at the tree level by

$$\Gamma^{(0)}_{\mu}(p, p-q) = Q\gamma_{\mu} \,. \tag{12.11}$$

where the superscript on Γ denotes the number of loops, and we have indicated the momenta of the incoming and outgoing fermions in parenthesis. Thus,

$$\begin{aligned} q^{\mu}\Gamma^{(0)}_{\mu}(p, p-q) &= Q q\!\!\!/ = Q\Big[(p\!\!\!/ - m) - (p\!\!\!/ - q\!\!\!/ - m)\Big] \\ &= Q\Big[S_F^{-1}(p) - S_F^{-1}(p-q)\Big]^{(0)} \,. \end{aligned} \tag{12.12}$$

This relation turns out to be valid at all orders in perturbation theory. In other words, we can write

$$q^{\mu}\Gamma_{\mu}(p, p-q) = Q\Big[S_F^{-1}(p) - S_F^{-1}(p-q)\Big] \tag{12.13}$$

as an exact result. This equality is called the *Ward-Takahashi identity*.

On the left hand side we have Γ_{μ}, which is the general 3-point function involving two fermions and one photon, as we have seen in

Ch. 11. The usefulness of this identity can be seen by interpreting the right hand side as the difference between 2-point functions, i.e., Feynman amplitudes of diagrams with two external legs. Let us first draw the total 2-point amplitude as the sum of the tree-level and the rest.

(12.14)

The tree-level amplitude, which is the undecorated line on the right hand side of Eq. (12.14), can be read off from the Lagrangian by treating the quadratic term just like an interaction and using the procedure for writing Feynman rules described in §6.6. This will give $\not{p} - m$, which is just the inverse of the tree-level propagator. The Feynman amplitude for the remainder, which is drawn as the gray box, is conventionally denoted by $-\Sigma(p)$. Thus Eq. (12.14) can be written algebraically as

$$S_F^{-1}(p) = \not{p} - m - \Sigma(p)\,. \tag{12.15}$$

It is this full 2-point function that goes into the Ward-Takahashi identity of Eq. (12.13).

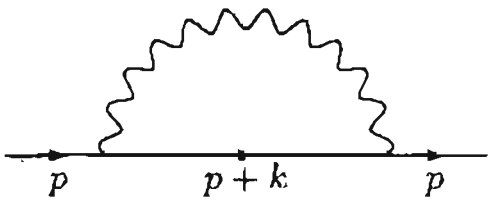


Figure 12.3: 1-loop diagram for fermion self-energy in QED.

To show that the Ward-Takahashi identity is at least valid for the 1-loop contributions, let us first evaluate the 1-loop corrections to the 2-point function for the fermions, i.e., the self-energy of the fermions. This comes from the diagram of Fig. 12.3. Using the Feynman rules of QED, we can write the 1-loop contribution to it as

$$-i\Sigma^{(1)}(p) = (-ieQ)^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{i}{\not{p} + \not{k} - m} \gamma_\nu \, iD^{\mu\nu}(k)\,, \tag{12.16}$$

where $iD^{\mu\nu}(k)$ is the photon propagator.

As for the vertex function, we have already seen the 1-loop contribution in Eq. (11.37). Although that was written for on-shell electrons, the expression remains the same except $-e$ is now replaced by eQ:

$$-ie\Gamma^{(1)}_{\mu} = (-ieQ)^3 \int \frac{d^4k}{(2\pi)^4}\gamma_\lambda \frac{i}{\not p' + \not k - m}\gamma_\mu \frac{i}{\not p + \not k - m}\gamma_\rho\, iD^{\lambda\rho}(k)\,, \tag{12.17}$$

where $p' = p - q$. Thus

$$q^\mu\Gamma^{(1)}_{\mu} = (-ieQ)^2 Q \int \frac{d^4k}{(2\pi)^4}\gamma_\lambda \frac{i}{\not p' + \not k - m}\not q \frac{i}{\not p + \not k - m}\gamma_\rho\, iD^{\lambda\rho}(k)\,. \tag{12.18}$$

Since $p' = p - q$, we can write $\not q$ as $(\not p + \not k - m) - (\not p' + \not k - m)$. Thus,

$$\frac{i}{\not p' + \not k - m}\not q \frac{i}{\not p + \not k - m} = i\left[\frac{i}{\not p' + \not k - m} - \frac{i}{\not p + \not k - m}\right]. \tag{12.19}$$

Putting this back into Eq. (12.18) and comparing with the expression for the self-energy in Eq. (12.16), we obtain

$$q^\mu\Gamma^{(1)}_{\mu}(p,p') = Q\left[\Big(-\Sigma^{(1)}(p)\Big) - \Big(-\Sigma^{(1)}(p')\Big)\right]. \tag{12.20}$$

Adding this to the tree level relation gives Eq. (12.13) up to 1-loop contributions.

It is true that our proof takes only the 1-loop contribution into account and considers only the fermion-photon interaction. But one can construct a more general proof from current conservation. In fact the identity holds true to all orders, even if the fermion has other interactions which do not violate gauge invariance. Also, we know from Eq. (11.8) that $q^\mu\Gamma_\mu$ must vanish between the spinors. However, if we consider the effective electromagnetic vertex of a neutral fermion for which $Q = 0$, we see from Eq. (12.13) that for neutral fermions a more stringent relation, $q^\mu\Gamma_\mu = 0$, is satisfied irrespective of whether we consider it between spinors or not. These facts are very useful in considering electromagnetic interactions of particles in the presence of other interactions as well.

Here we will stick to the electromagnetic interactions only. In view of the general identity of Eq. (12.13), we will consider below in detail only the loop contributions for self-energies of fermions and photons.

□ **Exercise 12.2** Suppose electrons also interact with a scalar field via a Yukawa interaction term $-h\bar{\psi}\psi\phi$. Show that at the 1-loop level, Eq. (12.20) still holds for the electron-photon vertex.

12.5 General forms for divergent amplitudes

12.5.1 Fermion self-energy

Let us start with the fermion self-energy in QED. We will take the fermion to be the electron, with $Q = -1$. The expression for the fermion self-energy can be written from Eq. (12.16) as

$$\begin{aligned}\Sigma(p) &= -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu(\not{p}+\not{k}+m)\gamma^\mu}{[(p+k)^2-m^2]k^2} \\ &= ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2(\not{p}+\not{k})-4m}{[(p+k)^2-m^2]k^2} \,. \end{aligned} \tag{12.21}$$

In the last step, we have used the contraction formulas for the γ-matrices, Eq. (4.42).

We can now use the Feynman parameters for the denominator, which were introduced in §11.3. Using Eq. (11.41), we write

$$\begin{aligned}\Sigma(p) &= ie^2 \int_0^1 d\zeta \int \frac{d^4k}{(2\pi)^4} \frac{2(\not{p}+\not{k})-4m}{[\zeta\{(p+k)^2-m^2\}+(1-\zeta)k^2]^2} \\ &= ie^2 \int_0^1 d\zeta \int \frac{d^4k}{(2\pi)^4} \frac{2(\not{p}+\not{k})-4m}{[k^2+2\zeta k\cdot p+\zeta(p^2-m^2)]^2} \,. \end{aligned} \tag{12.22}$$

Let us now define a new integration variable $\bar{k} = k + \zeta p$, so that we can write this as

$$\Sigma(p) = ie^2 \int_0^1 d\zeta \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{2(1-\zeta)\not{p}+2\not{\bar{k}}-4m}{\left[\bar{k}^2+\zeta(1-\zeta)p^2-\zeta m^2\right]^2} \,. \tag{12.23}$$

At this point, the denominator is even in k, so we can throw out the terms in the numerator which are odd in k. Thus we find that $\Sigma(p)$ has the form:

$$\Sigma(p) = a(p^2)\not{p} + b(p^2) \,, \tag{12.24}$$

where

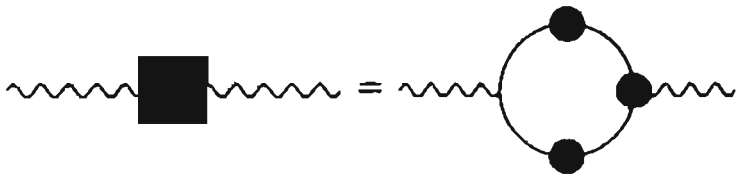
$$a(p^2) = 2ie^2 \int_0^1 d\zeta \int \frac{d^4k}{(2\pi)^4} \frac{(1-\zeta)}{[k^2+\zeta(1-\zeta)p^2-\zeta m^2]^2},$$
$$b(p^2) = -4mie^2 \int_0^1 d\zeta \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2+\zeta(1-\zeta)p^2-\zeta m^2]^2}. \quad (12.25)$$

We have dropped the bar on k. Of course, the form of these integrals show that both a and b are divergent. And this is the problem that we will have to deal with.

What we have done so far is to find the form of the fermion self-energy from the 1-loop diagram. However, a simple argument shows that no matter which diagram we consider, the general form of the self-energy will be given by Eq. (12.24) with suitably defined a and b. The self-energy function $\Sigma(p)$ will be a 4×4 matrix which can depend only on the 4-vector p. Now, any 4×4 matrix can be expanded in terms of the 16 basis matrices introduced in Eq. (4.14), which are the unit matrix, the four γ-matrices, the six $\sigma_{\mu\nu}$-matrices, the four matrices $\gamma^\mu\gamma_5$, and finally γ_5. Of these, the combinations involving γ_5 can never appear in QED because of parity invariance. With $\sigma_{\mu\nu}$, the only Lorentz covariant combination that one can make using the vector p^μ is $\sigma_{\mu\nu}p^\mu p^\nu$, which is zero because of the antisymmetry of the σ-matrices. Thus we are left with the unit matrix and the γ^μ's only. With the γ^μ's, the Lorentz covariant combination will be $\gamma^\mu p_\mu$, or $\not{p}$, which can appear with a co-efficient a, and the unit matrix can appear with a co-efficient b. Both these co-efficients must be Lorentz invariant, and can therefore depend only on p^2. This leads us to the form of Eq. (12.24).

12.5.2 Vacuum polarization

Let us denote the vacuum polarization amplitude by $\pi_{\mu\nu}(k)$, where the external photon momentum is k. The general form for the diagram has been shown in Fig. 12.2B. Consider the incoming photon line at the left. In any diagram, it must first encounter a vertex with the fermion. Thus, we can schematically represent the diagrams as



(12.26)

The blobs can be all possible combination of internal lines. The fermion line with the blob is the full fermion propagator, and the vertex with the blob is the full vertex function.

Therefore, the amplitude can be written as

$$i\pi_{\mu\nu}(k) = -\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[ieQ\gamma_\mu\, iS_F(p)\, ie\Gamma_\nu(k+p,p)\, iS_F(k+p)\right]. \tag{12.27}$$

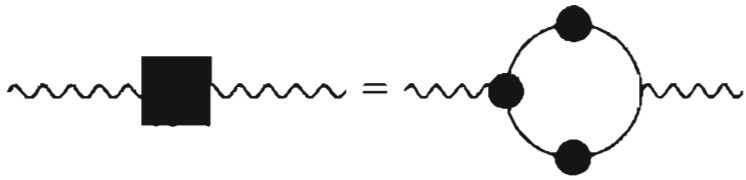
If we contract the left hand side of this equation by k^ν and use the Ward-Takahashi identity of Eq. (12.13), we obtain

$$k^\nu\pi_{\mu\nu}(k) = i(eQ)^2\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\gamma_\mu\Big(S_F(p) - S_F(k+p)\Big)\right]. \tag{12.28}$$

There are two terms here. In the second one, we can shift the loop integration variable from p to $k+p$, so that the two terms cancel each other and we obtain

$$k^\nu\pi_{\mu\nu}(k) = 0\,. \tag{12.29}$$

Similarly, starting from the diagrammatic identity



(12.30)

we can show that

$$k^\mu\pi_{\mu\nu}(k) = 0\,. \tag{12.31}$$

A small remark is in order. We have represented the vacuum polarization diagram in two ways, viz., in Eqs. (12.26) and (12.30). It must be understood that these are the most general characterizations of the vacuum polarization diagrams. For example, we cannot put a blob at both vertices, because this will double count diagrams like the one in Fig. 12.4.

The function $\pi_{\mu\nu}(k)$ must satisfy the conditions of Eqs. (12.29) and (12.31) and can depend only on the 4-momentum k. The only tensor which has these properties is $g_{\mu\nu}k^2 - k_\mu k_\nu$. Therefore, the

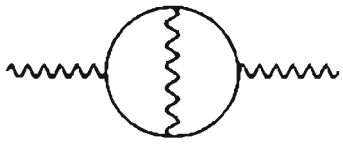


Figure 12.4: 2-loop vacuum polarization diagram which would be double counted if we put blobs at both vertices of Eq. (12.26) or Eq. (12.30).

most general form for the vacuum polarization tensor will be given by

$$\pi_{\mu\nu}(k) = \left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\Pi(k^2)\,, \tag{12.32}$$

where $\Pi(k^2)$ is a Lorentz invariant object, and must therefore be a function only of k^2. Eq. (12.32) vindicates the comment made earlier because the amplitude of the 2-point function with external photon lines is proportional to at least two powers of the external momentum.

□ **Exercise 12.3** Find the tensor structure for the effective 4-photon vertex $\pi_{\mu\nu\lambda\rho}(k_1, k_2, k_3, k_4)$. Assume parity invariance, and use relations like $k_1^\mu \pi_{\mu\nu\lambda\rho} = 0$, and the indistinguishability of photons which implies that the interchange between any two photons should not change the vertex. Do not assume the photons to be on-shell.

12.5.3 Vertex function

Finally, there is the divergence associated with the vertex function. The general form of the vertex function, denoted by Γ_μ, was discussed in Ch. 11. We found that in pure QED, the vertex can in general have two form factors. Of these, the anomalous magnetic moment term has an extra power of the external photon momentum in it, so the degree of divergence associated with this term should be -1. In other words, this term should be convergent, as we saw in the explicit 1-loop calculation of §11.3. The divergence, therefore, must be associated with the form factor F_1, i.e., with the charge form factor. This was explicitly shown in §11.4.

12.6 Regularization of self-energy diagrams

12.6.1 Vacuum polarization diagram

So far, we have not performed any regularization on the divergent amplitudes. In this section, let us show how regularization can be performed, taking the vacuum polarization diagram as an example. The lowest order loop diagram for vacuum polarization is shown in Fig. 12.5.

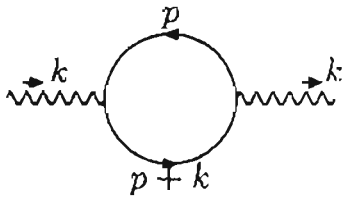


Figure 12.5: 1-loop diagram contributing to the vacuum polarization in QED.

As we mentioned in §12.3, we must introduce a regularization parameter in the theory in order to perform the integrals. We also mentioned some of the possibilities for the regularization parameter. In this section, let us use dimensional regularization. This means that all the integrals should be performed assuming the number of space-time dimensions to be an arbitrary real number N. We will also assume that the fermion in the loop is electron, so that its charge is $-e$.

Calculating in N dimensions poses a problem. The Lagrangian must now have mass dimension N. Using the Dirac Lagrangian, we find that the mass dimension of a fermion field is $[\psi] = (N-1)/2$. Similarly, from the Maxwell Lagrangian we find that $[A_\mu] = N/2-1$. Since the interaction term must also have mass dimension N, the coupling constant of the interaction term seems to have a non-zero mass dimension. We can take care of this by introducing an arbitrary constant mass μ and writing the interaction term as

$$\mathscr{L}_{\text{int}} = e\mu^{\varepsilon}\, \overline{\psi} \not{A} \psi \tag{12.33}$$

where e is dimensionless as usual, and we have defined

$$\varepsilon = 2 - \frac{1}{2}N\,. \tag{12.34}$$

This μ has the interpretation of a mass scale, and is usually called the *subtraction point*. We shall see that physical amplitudes and couplings are independent of this parameter.

Using the Feynman rules of QED, we can now write the amplitude of the vacuum polarization diagram of Fig. 12.5 in the form

$$i\pi_{\mu\nu}(k) = -\int \frac{d^N p}{(2\pi)^N} \text{ tr } \left[ie\mu^\varepsilon \gamma_\mu \frac{i(\not{p}+m)}{p^2-m^2} ie\mu^\varepsilon \gamma_\nu \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2} \right], \tag{12.35}$$

where the minus sign in front of the integral sign comes because of the closed fermion loop. Also, we now have $e\mu^\varepsilon$ instead of e at each vertex. This gives

$$\begin{aligned} \pi_{\mu\nu}(k) &= ie^2\mu^{2\varepsilon} \int \frac{d^N p}{(2\pi)^N} \frac{\text{tr } [\gamma_\mu \not{p} \gamma_\nu (\not{p}+\not{k}) + m^2 \gamma_\mu \gamma_\nu]}{[(p+k)^2-m^2][p^2-m^2]} \\ &= ie^2\mu^{2\varepsilon} \int \frac{d^N p}{(2\pi)^N} \\ &\quad \times \int_0^1 d\zeta \frac{\text{tr } [\gamma_\mu \not{p} \gamma_\nu (\not{p}+\not{k}) + m^2 \gamma_\mu \gamma_\nu]}{[\zeta\{(p+k)^2-m^2\} + (1-\zeta)(p^2-m^2)]^2}, \end{aligned} \tag{12.36}$$

introducing the Feynman parameter ζ in the last step. Changing the integration variable p to $p+\zeta k$, we can write this in the following form:

$$\begin{aligned} \pi_{\mu\nu}(k) &= ie^2\mu^{2\varepsilon} \int \frac{d^N p}{(2\pi)^N} \\ &\quad \times \int_0^1 d\zeta \frac{\text{tr } [\gamma_\mu \{\not{p}-\zeta\not{k}\} \gamma_\nu \{\not{p}+(1-\zeta)\not{k}\} + m^2 \gamma_\mu \gamma_\nu]}{[p^2-a^2]^2}, \end{aligned} \tag{12.37}$$

where

$$a^2 = m^2 - \zeta(1-\zeta)k^2 . \tag{12.38}$$

Let us now look at the trace in the numerator. At this point, the denominator is already even in the integration variable p, so we need only to keep the even terms in the numerator. Thus the numerator, which we will call $N_{\mu\nu}$, can be written as

$$N_{\mu\nu} = \text{tr } [\gamma_\mu \not{p} \gamma_\nu \not{p}] - \zeta(1-\zeta) \text{ tr } [\gamma_\mu \not{k} \gamma_\nu \not{k}] + m^2 \text{ tr } [\gamma_\mu \gamma_\nu] . \tag{12.39}$$

Such traces have been performed before in various calculations, but one should be careful here. The reason is that we are now supposed to perform the traces for N-dimensional space-time. So, let us say that, for N-dimensional space-time

$$\text{tr}\,[\gamma_\mu \gamma_\nu] = d_N g_{\mu\nu}\,, \tag{12.40}$$

where $d_N = 4$ when $N = 4$. Once this is assumed, we can find the trace of a string of four γ-matrices in the way described in Eq. (A.20) of the Appendix, and obtain

$$\text{tr}\,[\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho] = d_N\,(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho} + g_{\mu\rho}g_{\nu\lambda})\,. \tag{12.41}$$

Applying it to the present case, we get

$$\begin{aligned} N_{\mu\nu} &= d_N \left[(2p_\mu p_\nu - g_{\mu\nu}p^2) - \zeta(1-\zeta)(2k_\mu k_\nu - g_{\mu\nu}k^2) + m^2 g_{\mu\nu}\right] \\ &= -2d_N\zeta(1-\zeta)(k_\mu k_\nu - g_{\mu\nu}k^2) \\ &\qquad + d_N\left[2p_\mu p_\nu + g_{\mu\nu}\left(-p^2 + a^2\right)\right], \end{aligned} \tag{12.42}$$

using the definition of the quantity a^2 from Eq. (12.38).

For four dimensional space-time, we showed in Eq. (11.76) that the integral of $p_\mu p_\nu$ is the same as the integral of $\frac{1}{4}g_{\mu\nu}p^2$. In the same way, we can argue that in N-dimensional space-time, the integral of $p_\mu p_\nu$ is the same as the integral of $\frac{1}{N}g_{\mu\nu}p^2$. Therefore, the numerator can be rewritten as

$$N_{\mu\nu} = 2d_N\zeta(1-\zeta)(g_{\mu\nu}k^2 - k_\mu k_\nu) + d_N g_{\mu\nu}\left[\left(\frac{2}{N} - 1\right)p^2 + a^2\right]. \tag{12.43}$$

We can now perform the momentum integration. The general technique for this has been described in §A.4 of the Appendix. Let us first apply the results on the second term in the expression for the numerator. The co-efficient of $d_N g_{\mu\nu}$ in this integral is

$$\int_0^1 d\zeta \left[\left(\frac{2}{N} - 1\right) I_2^1(a,N) + a^2 I_2^0(a,N)\right], \tag{12.44}$$

where

$$I_s^r(a,N) \equiv \int \frac{d^N p}{(2\pi)^N}\,\frac{(p^2)^r}{(p^2 - a^2)^s}\,, \tag{12.45}$$

as defined in Eq. (A.45). Borrowing the result of this integration from Eq. (A.53), we can write the expression of Eq. (12.44) as

$$\frac{-i}{(4\pi)^{N/2}\Gamma(\frac{1}{2}N)\Gamma(2)}\int_0^1 d\zeta\left[\left(\frac{2}{N}-1\right)\frac{\Gamma(1+\frac{1}{2}N)\Gamma(1-\frac{1}{2}N)}{(a^2)^{1-N/2}} - a^2\frac{\Gamma(\frac{1}{2}N)\Gamma(2-\frac{1}{2}N)}{(a^2)^{2-N/2}}\right]. \quad (12.46)$$

Using now the property $\Gamma(1+z) = z\Gamma(z)$ which is valid for any z, we can write it as

$$\frac{-i}{(4\pi)^{N/2}}\int_0^1 d\zeta \;\; \frac{1}{(a^2)^{1-N/2}}\left[\frac{N}{2}\left(\frac{2}{N}-1\right)\Gamma(1-\frac{1}{2}N) - \Gamma(2-\frac{1}{2}N)\right]. \quad (12.47)$$

But the first term inside the square brackets is $(1-\frac{1}{2}N)\Gamma(1-\frac{1}{2}N) = \Gamma(2-\frac{1}{2}N)$. So the integral vanishes.

Actually, this was only expected. From the general grounds of gauge invariance, we have already argued in §12.5 that the tensor structure of $\pi_{\mu\nu}$ should be given by Eq. (12.32). And in writing the 1-loop expression for the numerator $N_{\mu\nu}$ in Eq. (12.43), we have already separated out the terms which correspond to this tensor structure. The remaining terms must then vanish, which is what we have shown explicitly. We can think of this as a cross-check on our implicit belief that our procedure of regularization does not destroy the gauge invariance of QED. We now go back to the expression in Eq. (12.43) and put it in the formula obtained in Eq. (12.37) to write

$$\pi_{\mu\nu}(k) = 2id_N e^2\mu^{2\epsilon}(g_{\mu\nu}k^2 - k_\mu k_\nu)\int\frac{d^Np}{(2\pi)^N}\int_0^1 d\zeta\;\frac{\zeta(1-\zeta)}{[p^2-a^2]^2}. \quad (12.48)$$

The momentum integration can be performed with the help of Eq. (A.53). This gives

$$\pi_{\mu\nu}(k) = -\frac{2d_N e^2\mu^{2\epsilon}}{(4\pi)^{N/2}}(g_{\mu\nu}k^2 - k_\mu k_\nu)\int_0^1 d\zeta\;\zeta(1-\zeta)\frac{\Gamma(\epsilon)}{(a^2)^\epsilon}. \quad (12.49)$$

We can now see why the result diverges for $N = 4$. For small ϵ,

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon), \quad (12.50)$$

where γ_E is the Euler-Mascheroni constant which is defined by

$$\gamma_E = -\int_0^\infty dx\; e^{-x} \ln x = 0.57721\cdots\,, \tag{12.51}$$

and the symbol $\mathcal{O}(\varepsilon)$ in Eq. (12.50) represent terms which have at least one power of ε, and therefore vanish for $\varepsilon = 0$. Using this expansion of the Γ-function with $d_N = 4$, we can write the dimension-dependent terms of Eq. (12.49) in the form:

$$\begin{aligned}\frac{\mu^{2\epsilon}}{4\pi^2}\Gamma(\varepsilon)\left(\frac{a^2}{4\pi}\right)^{-\varepsilon} &= \frac{1}{4\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)\right]\exp\left(-\varepsilon\ln\left(\frac{a^2}{4\pi\mu^2}\right)\right)\\ &= \frac{1}{4\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E\right]\left[1 - \varepsilon\ln\left(\frac{a^2}{4\pi\mu^2}\right)\right] + \mathcal{O}(\varepsilon)\\ &= \frac{1}{4\pi^2}\left[\frac{1}{\varepsilon} - \gamma_E - \ln\left(\frac{a^2}{4\pi\mu^2}\right)\right] + \mathcal{O}(\varepsilon)\,. \end{aligned}\tag{12.52}$$

Putting this back in Eq. (12.49) and using $\alpha = e^2/4\pi$, we can now write

$$\pi_{\mu\nu}(k) = -\frac{2\alpha}{\pi}(g_{\mu\nu}k^2 - k_\mu k_\nu)\left[\frac{1}{6\varepsilon'} - I(k^2)\right], \tag{12.53}$$

where we have used the shorthands

$$I(k^2) = \int_0^1 d\zeta\; \zeta(1-\zeta)\ln\left(\frac{m^2 - \zeta(1-\zeta)k^2}{\mu^2}\right), \tag{12.54}$$

reinstating the definition of a^2 from Eq. (12.38), and

$$\frac{1}{\varepsilon'} = \frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\,. \tag{12.55}$$

In the notation of Eq. (12.32),

$$\begin{aligned}\Pi(k^2) &= -\frac{2\alpha}{\pi}\left[\frac{1}{6\varepsilon'} - I(k^2)\right]\\ &= -\frac{\alpha}{3\pi}\left[\frac{1}{\varepsilon'} - \ln\left(\frac{m^2}{\mu^2}\right) + (\text{finite})k^2 + \cdots\right]\\ &= \Pi(0) + k^2\Pi'(0) + \cdots\,. \end{aligned}\tag{12.56}$$

This is as far as we can go with the regularization procedure. We introduced a regularization parameter N, or equivalently ε, as well as a mass scale μ, in order to have some control over the infinities. At the end, we see that the amplitude that we calculated depends on ε and μ. This is certainly unphysical, since these are unphysical parameters which we introduced into the theory. So we need to have some way of making the amplitude independent of these parameters. This is the process of renormalization, which will be discussed in §12.8. Before that, we want to take a look at the regularization of other divergent amplitudes of QED.

□ **Exercise 12.4** Show that in N-dimensions, the contraction formulas for the Dirac matrices take the forms

$$\begin{aligned} \gamma_\lambda \gamma^\lambda &= N = 4 - 2\varepsilon\,, \\ \gamma_\lambda \gamma_\mu \gamma^\lambda &= (-2 + 2\varepsilon)\gamma_\mu\,. \end{aligned} \tag{12.57}$$

12.6.2 Fermion self-energy diagram

For the vacuum polarization diagram, we used the dimensional regularization. We could use the same technique here. However, to give some idea of different regularization techniques, let us employ the Pauli-Villars regularization for this case. In this method, we introduce some extra particle in the theory whose mass is very heavy, to be denoted by M. This is the regularization parameter, with the physical limit corresponding to $M \to \infty$. We suppose that there is an extra diagram where this heavy particle can replace the photon. Also, the couplings of this heavy particle is such that the sign of this extra contribution is opposite to that of the photon diagram. Effectively this means that the denominator of the photon propagator should be modified. Instead of just k^2, we should now have

$$\frac{1}{k^2} \to \frac{1}{k^2} - \frac{1}{k^2 - M^2} = \frac{-M^2}{k^2(k^2 - M^2)}\,. \tag{12.58}$$

Therefore, instead of Eq. (12.21), we should now write

$$\Sigma(p) = ie^2 M^2 \int \frac{d^4k}{(2\pi)^4} \frac{-2(\not{p} + \not{k}) + 4m}{k^2(k^2 - M^2)[(p+k)^2 - m^2]}\,. \tag{12.59}$$

We can now introduce Feynman parameters for the denominator to write

$$\int \frac{d^4k}{(2\pi)^4} \frac{f(k)}{k^2(k^2-M^2)[(p+k)^2-m^2]}$$
$$= 2! \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \int_0^1 d\zeta_3 \, \frac{f(k)\delta(1-\zeta_1-\zeta_2-\zeta_3)}{D^3} . \quad (12.60)$$

Here

$$\begin{aligned} D &= \zeta_1((p+k)^2-m^2) + \zeta_2(k^2-M^2) + \zeta_3 k^2 \\ &= k^2 + 2\zeta_1 k\cdot p + \zeta_1(p^2-m^2) - \zeta_2 M^2 \\ &= (k+\zeta_1 p)^2 - C^2 , \end{aligned} \quad (12.61)$$

with

$$C^2 = \zeta_1 m^2 + \zeta_2 M^2 - \zeta_1(1-\zeta_1)p^2 , \quad (12.62)$$

and we have used $\zeta_3 = 1-\zeta_1-\zeta_2$. Putting this back into Eq. (12.59) and changing the integration variable from k to $k+\zeta_1 p$, we can write

$$\Sigma(p) = 2ie^2M^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\zeta_1 \int_0^{1-\zeta_1} d\zeta_2 \, \frac{-2(1-\zeta_1)\not{p} + 4m}{(k^2-C^2)^3} . \quad (12.63)$$

The parameter ζ_3 has been integrated over, and the resulting Θ-function has been used to change the limits of integration.

The momentum integration is now convergent, and we can evaluate it using Eq. (A.54). In the notation of Eq. (12.24), we find

$$\begin{aligned} a &= -\frac{\alpha M^2}{2\pi} \int_0^1 d\zeta_1 \, (1-\zeta_1) J(\zeta_1) , \\ b &= \frac{\alpha M^2 m}{\pi} \int_0^1 d\zeta_1 \, J(\zeta_1) , \end{aligned} \quad (12.64)$$

where

$$J(\zeta_1) = \int_0^{1-\zeta_1} d\zeta_2 \, \frac{1}{\zeta_1 m^2 + \zeta_2 M^2 - \zeta_1(1-\zeta_1)p^2} . \quad (12.65)$$

The integral over ζ_2 in Eq. (12.65) can be performed to give

$$\begin{aligned} J(\zeta_1) &= \frac{1}{M^2} \ln\left[\frac{(1-\zeta_1)M^2 + \zeta_1 m^2 - \zeta_1(1-\zeta_1)p^2}{\zeta_1 m^2 - \zeta_1(1-\zeta_1)p^2}\right] \\ &\approx \frac{1}{M^2} \ln\left[\frac{(1-\zeta_1)M^2}{\zeta_1 m^2 - \zeta_1(1-\zeta_1)p^2}\right] . \end{aligned} \quad (12.66)$$

In the last step, we have used the fact that M^2 is much larger than all the other mass or momentum scales in the problem. Putting this back in Eq. (12.64) and writing ζ for ζ_1, we obtain

$$a = \frac{\alpha}{2\pi}\int_0^1 d\zeta\,(1-\zeta)\ln\left[\frac{\zeta m^2 - \zeta(1-\zeta)p^2}{(1-\zeta)M^2}\right],$$
$$b = -\frac{\alpha m}{\pi}\int_0^1 d\zeta\,\ln\left[\frac{\zeta m^2 - \zeta(1-\zeta)p^2}{(1-\zeta)M^2}\right]. \qquad (12.67)$$

We see that the self-energy $\Sigma(p)$ depends on the regularization parameter M through a and b.

□ **Exercise 12.5** Starting from the first expression of Eq. (12.21), use dimensional regularization to evaluate the co-efficients a and b and obtain the results

$$a = \frac{\alpha}{2\pi}\left[-\frac{1}{2\varepsilon'} + \frac{1}{2} + \int_0^1 d\zeta\,(1-\zeta)\ln\left\{\frac{\zeta m^2 - \zeta(1-\zeta)p^2}{\mu^2}\right\}\right]$$
$$b = -\frac{\alpha m}{\pi}\left[-\frac{1}{\varepsilon'} + \frac{1}{2} + \int_0^1 d\zeta\,\ln\left\{\frac{\zeta m^2 - \zeta(1-\zeta)p^2}{\mu^2}\right\}\right]. \qquad (12.68)$$

[Hint: You must use the contraction formulas of Eq. (12.57).]

12.7 Counterterms

12.7.1 Vacuum polarization diagram

In §12.6.1, we calculated the amplitude for the 1-loop diagram for the vacuum polarization, and found the amplitude to be dependent on the regularization parameter. Other divergent amplitudes were also found to be dependent on the regularization parameter. All this was obtained by starting from the Lagrangian

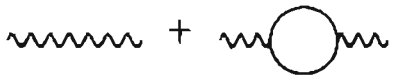
$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi + e\overline{\psi}\gamma_\mu\psi A^\mu, \qquad (12.69)$$

which is the Lagrangian of QED. There is also a gauge fixing term, as discussed in Ch. 8, but that is unimportant for the following discussion.

Let us consider for example the 2-photon amplitude. We can decompose it into a tree-level amplitude and the rest, as we did for the fermion in Eq. (12.14). The tree-level amplitude is given by

$$= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right), \qquad (12.70)$$

taking into account a symmetry factor of 2 which comes because A^ν, e.g., can refer to either of the two external lines. In addition, we have found the 1-loop contribution to the 2-photon amplitude in Eq. (12.53). Adding these two contributions, we can write



$$= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left\{1 - \Pi(k^2)\right\}$$
$$= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left\{1 + \frac{2\alpha}{\pi}\left[\frac{1}{6\epsilon'} - I(k^2)\right]\right\}. \qquad (12.71)$$

This is the correct expression for the 2-photon amplitude at order α that comes from the Lagrangian of Eq. (12.69).

The Lagrangian of Eq. (12.69) is the classical Lagrangian. This was useful in figuring out what the terms in the perturbation series would look like. But we cannot rule out quantum corrections to the Lagrangian which vanish as $\hbar \to 0$. So far, we have ignored $\hbar$ because of our choice of units. However, $\hbar$ can be thought of as a loop-counting parameter for the following reason.

The evolution operator given in Eq. (5.33) is dimensionless. When we restore units, we find a factor of $\hbar^{-1}$ with every H_I. So in calculating the amplitude of a loop, we will find one power of $\hbar^{-1}$ for each vertex. The free Hamiltonian also picks up a factor of $\hbar^{-1}$. Since the propagator is the inverse of the differential operator occurring in the quadratic terms in the Lagrangian, there will be a power of $\hbar$ for each propagator. So the total power of $\hbar$ for an ℓ-loop amplitude with n internal lines and v vertices is

$$n - v = \ell - 1\,, \qquad (12.72)$$

where we have used Eq. (6.56). In other words, $\frac{1}{\hbar}\mathscr{H}_{\text{eff}}$ has a contribution from ℓ-loop diagrams which is $\mathcal{O}(\hbar^\ell)$.

Therefore, when we talk about loop amplitudes, we are in fact talking about terms of higher order in $\hbar$ which vanish as $\hbar \to 0$. The full 'quantum' Lagrangian is therefore of the form

$$\mathscr{L} = \mathscr{L}_{\text{classical}} + \hbar\delta\mathscr{L}^{(1)} + \hbar^2\delta\mathscr{L}^{(2)} + \cdots, \qquad (12.73)$$

where $\delta\mathscr{L}^{(n)}$ will affect the results at n-loop order. For example, when we calculated the propagator to 1-loop order in Eq. (12.71),

we ought to have included some unspecified $\delta\mathscr{L}^{(1)}$ in Eq. (12.69). Let us include such a term, and add its contribution to the result of Eq. (12.71). This will give the full amplitude at order $\hbar$. Let us denote the contribution of $\delta\mathscr{L}^{(1)}$ diagrammatically by a cross on the photon line. All we know about $\delta\mathscr{L}^{(1)}$ is that $\hbar\delta\mathscr{L}^{(1)}$ vanishes as $\hbar \to 0$. Other than that we are free to choose its form to be whatever is convenient for us. Let us suppose its contribution is given by

$$= \left(g_{\mu\nu}k^2 - k_\mu k_\nu\right) \frac{2\alpha}{\pi} \frac{1}{6} \left(\frac{1}{\epsilon'} - \ln\frac{m^2}{\mu^2}\right). \quad (12.74)$$

If we now include this contribution, the total contribution to the 2-photon amplitude up to order α is given by

$$= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right) \times \left\{1 - \frac{2\alpha}{\pi}\int_0^1 d\zeta\, \zeta(1-\zeta)\ln\left(1 - \zeta(1-\zeta)\frac{k^2}{m^2}\right)\right\}. \quad (12.75)$$

There are two things to note here. First, the dependence on the regularization parameter has vanished. For this reason, this new term is called a *counterterm*, i.e., something which counters the dependence of a divergent amplitude on the regularization parameter. Second, for a physical photon, $k^2 = 0$, the logarithm term in the integrand vanishes, so that the contribution for the 2-photon amplitude is exactly what we obtained from the tree diagram. These are the two criteria which led us to choose the precise form of the counterterm contribution of Eq. (12.74). Formally, we define the renormalized vacuum polarization through

$$\Pi_R(k^2) = \Pi(k^2) + \Pi_{CT}\,, \quad (12.76)$$

and fix Π_{CT} by the on-shell renormalization prescription

$$\Pi_R(0) = 0\,. \quad (12.77)$$

For arbitrary k, the 2-point function now becomes

$$-\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\left\{1 - \Pi_R(k^2)\right\}. \quad (12.78)$$

What is the counterterm which will give the contribution of Eq. (12.74)? This is easy. In fact, we can use a counterterm Lagrangian which is

$$\mathscr{L}_{\rm CT}^{(B)} = \frac{1}{4}\,\frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon'} - \ln\frac{m^2}{\mu^2}\right) F_{\mu\nu}F^{\mu\nu} \tag{12.79}$$

This, of course, is the proper form for the counterterm only at the order α. If more complicated diagrams for vacuum polarization are discussed, including terms of higher order in α, the precise form for the counterterm Lagrangian will be different. However, since the divergent contributions to $\pi_{\mu\nu}$ always have the same tensor structure as in Eq. (12.32), the counterterm will in general be of the form

$$\mathscr{L}_{\rm CT}^{(B)} = -\frac{1}{4}\,(Z_3 - 1)F_{\mu\nu}F^{\mu\nu}\,, \tag{12.80}$$

for some Z_3. At 1-loop, our choice corresponds to

$$Z_3 = 1 - \frac{\alpha}{3\pi}\left(\frac{1}{\varepsilon'} - \ln\frac{m^2}{\mu^2}\right). \tag{12.81}$$

12.7.2 Fermion self-energy diagram

In the fermion self-energy, we found that there is a divergent part which goes like $\not{p}$, and there is another one which goes like the unit matrix. Thus, the divergences in these can be canceled by a counterterm Lagrangian of the form

$$\mathscr{L}_{\rm CT}^{(A)} = (Z_2 - 1)\overline{\psi}\,(i\gamma^\mu\partial_\mu - (m - \delta m))\,\psi\,, \tag{12.82}$$

for some Z_2 and δm. We want this term to counter the divergences coming from fermion self-energy. The total inverse propagator would now be

$$\not{p} - m - \Sigma_R(p)\,, \tag{12.83}$$

where $\Sigma_R(p)$ is the sum of the divergent loop contributions and the counterterms,

$$\Sigma_R(p) = \Big(a(p^2) + a_{\rm CT}\Big)\not{p} + \Big(b(p^2) + b_{\rm CT}\Big)\,. \tag{12.84}$$

Comparing this equation with Eq. (12.82), we see that $Z_2 - 1 = -a_{\rm CT}$ and $(Z_2 - 1)(m - \delta m) = b_{\rm CT}$.

We now fix $a_{\rm CT}$ and $b_{\rm CT}$ by the renormalization prescription

$$\Sigma_R(p)\Big|_{\not p=m} = 0\,,$$
$$\frac{\partial\Sigma_R(p)}{\partial\not p}\Big|_{\not p=m} = 0\,. \tag{12.85}$$

This gives

$$\begin{aligned} a_{\rm CT} &= -a(m^2) - 2m^2a'(m^2) - 2mb'(m^2)\,,\\ b_{\rm CT} &= -b(m^2) + 2m^3a'(m^2) + 2m^2b'(m^2)\,, \end{aligned} \tag{12.86}$$

where a', b' are derivatives with respect to p^2. After some algebra, we get

$$\begin{aligned} a_{\rm CT} &= -\frac{\alpha}{2\pi}\int_0^1 d\zeta\,(1-\zeta)\left[\ln\left(\frac{\zeta^2m^2}{(1-\zeta)M^2}\right) + \frac{2(1+\zeta)}{\zeta}\right],\\ b_{\rm CT} &= \frac{\alpha m}{\pi}\int_0^1 d\zeta\,\left[\ln\left(\frac{\zeta^2m^2}{(1-\zeta)M^2}\right) + \frac{1-\zeta^2}{\zeta}\right]. \end{aligned} \tag{12.87}$$

If we now substitute $a(p^2)$, $b(p^2)$ from Eq. (12.67) and $a_{\rm CT}$, $b_{\rm CT}$ from Eq. (12.87), we find that Σ_R is independent of the regularization parameter M. Since the $M\to\infty$ limit gives the ultra-violet divergences, we can say that after adding the counterterms, the ultra-violet divergence has vanished in the 2-point function of Eq. (12.83).

This is not to say that Eq. (12.87) is free from divergences altogether. It involves the integral $\int_0^1 d\zeta/\zeta$, which clearly diverges at the lower limit. This divergence can be traced back to Eq. (12.23). Near $\zeta = 0$, the momentum integral is $\int d^4k/k^4$, which diverges at both the infra-red and ultra-violet regions. The counterterm has removed the ultra-violet divergence, but the infra-red divergence remains. We will show in §12.9.3 that this infra-red divergence cancels in physical amplitudes.

The on-shell criterion for choosing our counterterms guarantees that divergent decorations of the external (on-shell) lines in a 1PR diagram can be safely ignored, because they cancel with the counterterm.

□ **Exercise 12.6** Supply the missing algebra between Eqs. (12.85) and (12.87). For arbitrary external momentum p, find the 2-point function up to order α by adding the tree-level result, the 1-loop result, and the counterterm.

□ **Exercise 12.7** Use the dimensional regularization technique, for which the 1-loop results for the a and the b co-efficients have been given in Eq. (12.68). Now choose the counterterms by the same prescription as in Eq. (12.85). Show that after adding the counterterms, the expression obtained for the 2-point function with fermions is the same as the one obtained with the Pauli-Villars regularization scheme.

12.7.3 Vertex function

The remaining divergence is in the vertex amplitude. It can be canceled by a counterterm of the form

$$\mathscr{L}_{\rm CT}^{(C)} = (Z_1 - 1)e\overline{\psi}\gamma_\mu\psi A^\mu\,, \tag{12.88}$$

for some Z_1. Notice that although the vertex can have anomalous magnetic moment kind of terms, those terms are not divergent, so one does not need any counterterm for them.

The counterterms are related by the Ward-Takahashi identity derived in §12.4. To show this, let us write the vertex function for the electron as

$$\Gamma_\mu(p,p') = -\gamma_\mu + \Lambda_\mu(p,p')\,, \tag{12.89}$$

where $\Lambda_\mu(p,p')$ is the loop contribution. Taking an infinitesimal q in Eq. (12.13), we can write the identity in the form

$$\Lambda_\mu(p,p) = -\frac{\partial\Sigma(p)}{\partial p^\mu}\,, \tag{12.90}$$

which is the original form in which the identity was derived by Ward. In principle, we can introduce counterterms so that this relation is violated at higher loops. But suppose we design the renormalization prescriptions in such a way that Eq. (12.90) is obeyed for the corresponding counterterms as well. From Eq. (12.82), we see that

$$-\Sigma_{\rm CT} = (Z_2 - 1)\not{p} + \text{terms independent of } p\,, \tag{12.91}$$

so that

$$-\frac{\partial\Sigma_{\rm CT}}{\partial p^\mu} = (Z_2 - 1)\gamma_\mu\,. \tag{12.92}$$

On the other hand, Eq. (12.88) gives

$$\Lambda_\mu^{\rm CT} = (Z_1 - 1)\gamma_\mu\,, \tag{12.93}$$

so that we obtain

$$Z_1 = Z_2 \,. \tag{12.94}$$

12.8 Full Lagrangian

We have seen that if we start from the classical Lagrangian and calculate amplitudes for various physical processes, we encounter infinities. So we need to add counterterms to the classical Lagrangian of Eq. (12.69). The general form of these counterterms were given in Eqs. (12.80), (12.82) and (12.88). The full Lagrangian would be obtained by adding these counterterms to the original Lagrangian. This gives

$$\begin{aligned}\mathscr{L}_B &= \mathscr{L} + \mathscr{L}_{\rm CT} \\ &= -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + Z_2 \overline{\psi} \left(i\gamma^\mu \partial_\mu - m_B \right) \psi + Z_1 e \overline{\psi} \gamma_\mu \psi A^\mu ,\end{aligned} \tag{12.95}$$

where

$$m_B = m - (1 - Z_2^{-1}) \delta m \,. \tag{12.96}$$

Suppose now we define a new set of fields, represented by a script index B (for 'bare'), by the relations

$$A_B^\mu = \sqrt{Z_3}\, A^\mu \,, \qquad \psi_B = \sqrt{Z_2}\, \psi \,. \tag{12.97}$$

What we find is that in terms of these new fields, the full or bare Lagrangian can be written in the form

$$\mathscr{L}_B = -\frac{1}{4} F_{B\mu\nu} F_B^{\mu\nu} + \overline{\psi}_B \left(i\gamma^\mu \partial_\mu - m_B \right) \psi_B + e_B \overline{\psi}_B \gamma_\mu \psi_B A_B^\mu \,, \tag{12.98}$$

where

$$e_B = \frac{Z_1}{Z_2 \sqrt{Z_3}}\, e \,. \tag{12.99}$$

The interesting thing is that the Lagrangian of Eq. (12.98) looks exactly like the Lagrangian of Eq. (12.69), except with a different

set of fields and a different set of parameters, like m_B and e_B in place of m and e. All we have done is to change the normalization of the fields, a process which can aptly be called *renormalization.* We have not explicitly written down the gauge fixing term, but the same comment applies to it as well. The parameters m_B and e_B contain infinite contributions, but physical quantities like amplitudes depend on the finite physical parameters m and e.

Another way of saying the same thing is that, all the counterterms can be hidden in the definitions of the fields and the parameters in the full Lagrangian. This is the essence of the renormalization procedure.

We could ask what would happen if we perform all calculations starting from the bare Lagrangian of Eq. (12.98), or equivalently of Eq. (12.95). In this case, we will not have any counterterm. The tree amplitudes for, say, the 2-point function of the photon or the fermion would be infinite in this case, since the bare parameters and bare fields are infinite. However, once we add the loop diagrams, there will be divergences coming from them as well. These divergences will precisely cancel those present in the tree amplitude and we will obtain a finite result. These are results which are compared with experiments.

Since the physical results can be obtained from the full Lagrangian, the bare parameters cannot depend on the renormalization prescriptions. This may sound odd since we have constructed the full Lagrangian by adding counterterms to the classical one. However, the division of the full Lagrangian into a classical part and the counterterms is no more than a choice. For example, in the vacuum polarization amplitude, we chose the counterterm to cancel the loop contribution at $k^2 = 0$. We could have chosen some non-zero value of k^2 to calculate the counterterm without affecting the physical results.

Coming back to Eq. (12.98) we see that after adding counterterms, the operator corresponding to the gauge covariant derivative is

$$D_\mu^{(B)} = \partial_\mu - ie_B A_{B\mu} = \partial_\mu - i\frac{Z_1}{Z_2} eA_\mu \,, \tag{12.100}$$

where we have used Eqs. (12.97) and (12.99). So we see that as long as the counterterms are chosen to satisfy the Ward-Takahashi identity, which implies $Z_1 = Z_2$, the gauge covariant derivative is unaffected by quantization, $D_\mu^{(B)} = D_\mu$. In other words, although

we have quantized the theory by adding gauge fixing terms, all other terms obey the gauge symmetry even in the full quantum theory.

We have gone through the procedure for the specific theory of QED. But the general philosophy gives some insight into all field theories. For example, we understand why theories in which some coupling constants have negative mass dimensions are non-renormalizable, a statement that we made earlier. As we discussed, in such theories, infinite number of amplitudes become divergent. Thus, to tame these divergences, we would need an infinite number of counterterms. But the original Lagrangian does not have infinite number of fields or parameters. Thus there is no way that we could hide the infinitely many counterterms in the definitions of fields and parameters of the theory, and this is equivalent to the statement that such theories are non-renormalizable.

12.9 Observable effects of renormalization

The renormalization procedure described above might seem only like a mathematical trick to hide the infinities. It is not so. In fact, it has observable consequences, some of which we now discuss.

12.9.1 Modification of Coulomb interaction

Classically, the interaction between two point charges is governed by the inverse square law. A small test charge would scatter off a fixed point charge according to the Rutherford scattering formula. We have already seen that treating the electron and the photon field as quantum objects leads to a modification of the scattering formula at the lowest order. However, there are also higher order terms in the S-matrix which contribute to this process, and they have to be included for high-precision results. The higher order diagrams will require renormalization, i.e., introduction of counterterms. But the numerical value of the counterterm depends on how many orders we include, and the energy of the particle involved.

The corrections to the photon propagator are shown schematically in Fig. 12.6. We have already calculated this correction at first order in α. For small momentum $k^2 \ll m^2$, we can take the inverse

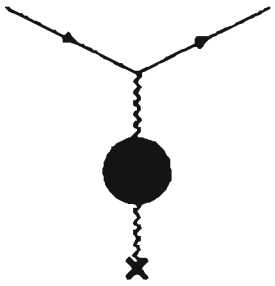


Figure 12.6: Correction to the static field in the scattering of electron by a nucleus of charge $+Ze$.

of Eq. (12.75) to obtain

$$D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2}\left(1 - \frac{\alpha}{15\pi}\frac{k^2}{m^2}\right) + \text{gauge terms}. \qquad (12.101)$$

This will affect the scattering of an electron off a static charge. Separating this propagator into its transverse, Coulomb and 'remainder' (gauge-dependent) parts as in §8.7, we find that the correction corresponds to replacing the static field of the nucleus by

$$\frac{Ze}{4\pi r} + \frac{Ze\alpha}{15\pi m^2}\,\delta^3(\boldsymbol{x})\,. \qquad (12.102)$$

Because of the δ-function, this extra term contributes only to the $l = 0$ states of an atom. As a result, it contributes to the energy difference between the $2s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}}$ levels, called Lamb shift.

12.9.2 Running coupling constant

Let us go back to Eq. (12.99). Since we now know that Ward identity gives $Z_1 = Z_2$, we can rewrite it as

$$e_B = \frac{e}{\sqrt{Z_3}}\,. \qquad (12.103)$$

In Eq. (12.81), the expression for Z_3 was derived assuming that the counterterms to the photon propagators should vanish on-shell. While defining the electric charge, however, we usually use the interaction strength between on-shell fermions. As shown in Ch. 9, the photon cannot be on-shell at a QED vertex if the fermions are.

So one needs to put fresh conditions to define the physical charge. The physical quantities appear in the Lagrangian of Eq. (12.69).

Suppose someone decides that the quantity e appearing in this Lagrangian is the value of the coupling constant at some specific value of k^2, where k is the momentum flowing through the photon line. For the interaction term, this person would then choose Z_3 in such a way that the counterterm cancels the divergent contributions at that value of k^2.

As for the possible values of k^2, we note one thing. Suppose the two fermion lines connected with the photon line carry momenta p and p'. Then $k^2 = (p - p')^2 = 2(m^2 - p \cdot p')$. If we consider it in the rest frame of the initial particle, this gives $k^2 = 2m(m - E')$, where E' is the time component of p'. Since $E' > m$, this quantity is always negative. And since k^2 is a Lorentz invariant quantity, it must be negative in every frame.

So let us say we decided that the quantity e appearing in Eq. (12.69) is the value of the coupling constant at $k^2 = -s$, where now s is a positive quantity. For this choice, let us write e as $e(s)$, and the resulting choice of Z_3 will be denoted by $Z_3(s)$. Looking back at the sum of the tree contribution and the 1-loop divergent contribution in Eq. (12.71), we now see that we should choose

$$Z_3(s) = 1 - \frac{2\alpha}{\pi}\left[\frac{1}{6\epsilon'} - I(-s)\right] . \tag{12.104}$$

Thus to order α in which these counterterms have been calculated, this exercise would yield

$$\alpha_B = \frac{\alpha(s)}{Z_3(s)} = \alpha(s)\left(1 + \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon'} - 6I(-s)\right]\right) , \tag{12.105}$$

ignoring terms which are higher order in α.

This equation has very interesting consequences. The quantity α_B is defined through e_B, which appears in the bare Lagrangian. So α_B does not depend on the scale at which we want to define our physical coupling constant e. However, since $Z_3(s)$ depends on s, the conclusion is that $\alpha(s)$ must also depend on s. The physical coupling constant depends on the momentum scale at which the interaction is taking place, and changes when the scale changes. Usually, this phenomenon is described by saying that the physical coupling constant is *running*.

To see this running explicitly, we eliminate the bare coupling α_B from Eq. (12.105). For this, let us consider another person who wants

to define the physical coupling constant appearing in Eq. (12.69) by the value of the coupling for $k^2 = -s'$. Then this person would obtain an equation which is obtained by replacing s by s' in Eq. (12.105). Equating now the two expressions for α_B, we obtain

$$\alpha(s) = \alpha(s') + \frac{2\alpha^2}{\pi}\left[I(-s) - I(-s')\right] . \tag{12.106}$$

It is to be noted that, although we are distinguishing between $\alpha(s)$ and $\alpha(s')$, we do not maintain the distinction while we write the α^2 term. This is because our expressions are correct only so far as the $\mathcal{O}(\alpha)$ corrections are concerned. If we want to distinguish the $\mathcal{O}(\alpha^2)$ terms, we need the expressions correct to $\mathcal{O}(\alpha^2)$, which can be obtained only from a 2-loop calculation. As far as our 1-loop calculation goes, Eq. (12.106) shows the correct accuracy of our result. It already shows that $\alpha(s)$ varies with s, but to make the nature of the variation explicit, let us consider two interesting limits.

Case 1: $s \gg m^2$, $s' \gg m^2$: In this case, we can neglect m^2 in the definition of $I(s)$ in Eq. (12.54) and get

$$\begin{aligned} I(-s) &\approx \int_0^1 d\zeta\, \zeta(1-\zeta) \ln\left(\frac{\zeta(1-\zeta)s}{\mu^2}\right) \\ &= \frac{1}{6}\ln\left(\frac{s}{\mu^2}\right) + \int_0^1 d\zeta\, \zeta(1-\zeta)\ln\left(\zeta(1-\zeta)\right) . \end{aligned} \tag{12.107}$$

Putting this back into Eq. (12.106), we obtain

$$\alpha(s) = \alpha(s') + \frac{\alpha^2}{3\pi}\ln\left(\frac{s}{s'}\right) . \tag{12.108}$$

Notice that the effective α increases with increasing values of s. This is why higher order diagrams are needed at higher energies in order to obtain the same precision.

Case 2: $s \gg m^2$, $s' = 0$: In this case we need the value of $I(0)$, which is

$$I(0) = \int_0^1 d\zeta\, \zeta(1-\zeta)\ln\left(\frac{m^2}{\mu^2}\right) = \frac{1}{6}\ln\left(\frac{m^2}{\mu^2}\right) . \tag{12.109}$$

Therefore,

$$I(-s) - I(0) = \frac{1}{6}\ln\left(\frac{s}{m^2}\right) + \int_0^1 d\zeta\, \zeta(1-\zeta)\ln\left(\zeta(1-\zeta)\right)$$
$$= \frac{1}{6}\left[\ln\left(\frac{s}{m^2}\right) - \frac{5}{3}\right]. \qquad (12.110)$$

Thus,

$$\alpha(s) = \alpha(0) + \frac{\alpha^2}{3\pi}\left[\ln\left(\frac{s}{m^2}\right) - \frac{5}{3}\right]. \qquad (12.111)$$

At $s = (80\,\text{GeV})^2$, corresponding to the mass of the W-boson, taking m to be the mass of the electron, and using $\alpha(0) = 1/137.03$, we find $\alpha(s) = 1/134.7$ from this formula. In our calculations we included only the contribution for electron loops. But there are other charged particles with masses below 80 GeV, which include the muon (106 MeV) and the tau (1.78 GeV). There are also the quarks u (5 MeV), d (9 MeV), s (170 MeV), c (1.4 GeV) and b (4.4 GeV) below 80 GeV. The masses shown here are upper limits of their estimates. Loops involving them will also contribute to the calculation. The effective coupling constant at 80 GeV turns out to be about 1/128. This is the value that should be used in calculations of QED at energies around 80 GeV.

□ **Exercise 12.8** Use the masses of the fermions given above to calculate α at $s = (80\,GeV)^2$.

12.9.3 Cancellation of infra-red divergences

In §9.10, we discussed the cross section for the Bremsstrahlung process in the field of a heavy static charge. We showed that the differential cross section calculated from the lowest order diagram suffers from an infra-red divergence. This would mean that we obtain an infinite number of very low energy photons in the radiation.

The resolution of this problem is a vindication of quantum field theory. Quantum physics, roughly speaking, is all about measurements. When making experimental observations on Bremsstrahlung, we can set up an experiment that records the final electron as well as the final photon. However, no matter how good our detector is, some very low energy photons will always go through without detection.

In other words, a process with no photon emission is experimentally indistinguishable from a process which emits a very low energy photon. Therefore in our calculations of low energy photon emission, we should include processes where no photon is emitted.

Such a process was already shown in Fig. 9.8, and its Feynman amplitude was denoted by $\mathscr{M}_0$ in §9.10. However, it cannot cancel the infra-red divergence of the process with soft photon emission by itself for two reasons. First, $\mathscr{M}_0$ does not have any infra-red divergence. Second, it has one less power of the coupling constant e compared to the Feynman amplitude for Bremsstrahlung, as seen from Eq. (9.136).

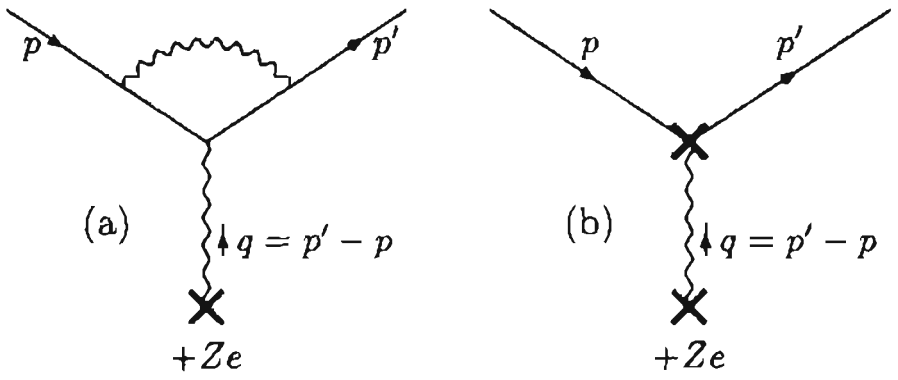


Figure 12.7: These processes cannot be distinguished from Bremsstrahlung with a soft final photon, so they must be included. The first one is a loop correction to the vertex function, the second one involves the vertex counterterm.

We now consider the next higher order term for the scattering process with no photon emission, shown in Fig. 12.7. The amplitude of these diagrams are $\mathcal{O}(e^2\mathscr{M}_0)$. When we add this with the lowest order amplitude $\mathscr{M}_0$ and square it, the cross term will have the same power of e as the square of the Bremsstrahlung amplitude. We now show that this cross term will also be infra-red divergent, and the two divergences exactly cancel each other.

We start with Fig. 12.7a. The expression for the vertex function is given in Eq. (11.38). For soft photons $k^\mu \approx 0$, so we neglect all occurrences of k in the numerator. Simplifying the denominator with the on-shell conditions $p^2 = p'^2 = m^2$ and putting in the $i\varepsilon$ terms in

the propagators, we can write it as

$$\Gamma^{(1)}_\mu = ie^2 \int_{\rm IR} \frac{d^4k}{(2\pi)^4} \frac{\gamma_\lambda(\not{p}' + m)\gamma_\mu(\not{p} + m)\gamma^\lambda}{[2p'\cdot k + i\varepsilon]\,[2p\cdot k + i\varepsilon]\,[k^2 + i\varepsilon]} + \cdots . \tag{12.112}$$

In this formula, the integral is not over all values of k, but rather only on small values, indicated by the subscript 'IR' on the integration sign. As in §9.10, the dots stand for the contribution from other values of k.

Using the contraction formulas of Eq. (4.42) and remembering that this entire expression is sandwiched between $\overline{u}(p')$ and $u(p)$, the numerator can be written as $4p\cdot p'\gamma_\mu$. So we obtain

$$\Gamma^{(1)}_\mu = ie^2 p\cdot p'\gamma_\mu \int_{\rm IR} \frac{d^4k}{(2\pi)^4} \frac{1}{p'_0k_0 - \boldsymbol{p}\cdot\boldsymbol{k} + i\varepsilon} \frac{1}{p_0k_0 - \boldsymbol{p}\cdot\boldsymbol{k} + i\varepsilon} \times \frac{1}{k_0^2 - \boldsymbol{k}^2 + i\varepsilon} + \cdots . \tag{12.113}$$

There are four poles of k_0 in the integrand. Let us choose the contour of integration to enclose the upper half plane, so that it will contain only one of them, viz., $k_0 = -\omega + i\varepsilon$, where $\omega = \mathbf{k}$. The result of the k_0-integration will be $2\pi i$ times the residue at this pole, so

$$\Gamma^{(1)}_\mu = -e^2\gamma_\mu \int_{\rm IR} \frac{d^3k}{(2\pi)^3} \frac{p\cdot p'}{(p'\cdot\tilde{k})(p\cdot\tilde{k})(-2\omega)} + \cdots , \tag{12.114}$$

where $\tilde{k}^\mu = (\omega, -\boldsymbol{k})$. Changing now the integration variables from k^i to $-k^i$, we obtain

$$\Gamma^{(1)}_\mu = \frac{\alpha}{(2\pi)^2}\gamma_\mu \int_{\rm IR} \frac{d^3k}{\omega} \frac{p\cdot p'}{(p'\cdot k)(p\cdot k)} + \cdots , \tag{12.115}$$

where now $k^\mu = (\omega, \boldsymbol{k})$. The Feynman amplitude for Fig. 12.7a is then

$$\overline{u}(p')\Big[-ie\Gamma^{(1)}_\mu\Big]u(p)A^\mu_e = -\frac{\alpha}{(2\pi)^2}\mathcal{M}_0 \times \int_{\rm IR} \frac{d^3k}{\omega} \frac{p\cdot p'}{(p'\cdot k)(p\cdot k)} + \cdots . \tag{12.116}$$

We now need to add the counterterm diagram given in Fig. 12.7b. The vertex counterterm is simply $-\Gamma_\mu$ for $q = 0$, i.e., $p = p'$. Putting

$p = p'$ in Eq. (12.116), we obtain the infra-red divergent part of the Feynman amplitude of this diagram to be

$$\frac{\alpha}{(2\pi)^2}\mathscr{M}_0 \int_{\rm IR} \frac{d^3k}{\omega}\,\frac{p^2}{(p\cdot k)^2}\,. \tag{12.117}$$

Performing the angular integrations, it is easy to see that the result does not depend on p, so we can write it equivalently as

$$\frac{\alpha}{2(2\pi)^2}\mathscr{M}_0 \int_{\rm IR} \frac{d^3k}{\omega}\left[\frac{p^2}{(p\cdot k)^2}+\frac{p'^2}{(p'\cdot k)^2}\right]. \tag{12.118}$$

Adding the contributions of Eqs. (12.116) and (12.118) with that of the lowest order scattering diagram, we obtain the amplitude for diagrams with no photon emission to be

$$\mathscr{M}_0' = \left[1+\frac{\alpha}{2(2\pi)^2}\int_{\rm IR}\frac{d^3k}{\omega}\left(\frac{p}{p\cdot k}-\frac{p'}{p'\cdot k}\right)^2+\cdots\right]\mathscr{M}_0\,. \tag{12.119}$$

When we square this, the cross term exactly cancels the infra-red divergent contribution in the Bremsstrahlung amplitude given in Eq. (9.138). Of course the $\mathcal{O}(\alpha^2)$ term appearing in $|\mathscr{M}_0'|^2$ is infra-red divergent as well. But to see the cancellation of that, we need to consider terms of higher order in α for the Bremsstrahlung amplitude. The part of the integral written as dots will contribute to finite 1-loop corrections, and to cancellation of ultra-violet divergences, both of which we have already seen.

The counterterms were introduced to eliminate the ultra-violet divergences. But we now see that they are required to cancel the infra-red divergences. We have used only the vertex counterterm for this calculation. This is because the fermions are on-shell, therefore fermion self-energy counterterms cancel exactly with the self-energy loop diagrams. The photon is not on-shell here, but its counterterm has no infra-red divergence and so is irrelevant for this calculation.

We have done the analysis for scattering from an external source and the associated Bremsstrahlung. But a similar analysis can be performed for the general situation where the infra-red divergence of soft photon emission from external charged particles in any process is canceled by the infra-red divergence of soft virtual photon exchange between external lines. Further it can be shown that the cancellation is effective at all orders of perturbation theory.

Chapter 13

Symmetries and symmetry breaking

At various points in this book, we have talked about different kinds of symmetries in field theories. Symmetries have important consequences on any quantum theory. They can relate masses or energies of different states, or scattering cross sections of different processes. For field theories in general, we discussed an important consequence of continuous symmetries in Ch. 2, viz., the Noether theorem. In this chapter, we will discuss some other general issues pertaining to symmetries in quantum field theories.

13.1 Classification of symmetries

Symmetries can be space-time symmetries or internal symmetries. In the first case, the symmetry transformations connect fields at different space-time points in general, as for example in the case of Lorentz transformations given in Ch. 2. In the other, the transformations connect different fields, or different components of the same field, at the same space-time point. An example is the gauge transformation of QED given in Ch. 9.

Symmetries can be discrete, where the parameter for the symmetry transformation can take only discrete values. Examples of such symmetries are parity, time reversal, charge conjugation, as discussed in Ch. 10. On the other hand, symmetries can also be continuous, where the relevant parameter (or parameters) can take continuous values. Example of such symmetries are invariances of various free

Lagrangians under phase rotations of the fields involved.

Among continuous symmetries, we have discussed two important classes. Symmetries can be global, where the transformation parameters are independent of space-time. The phase symmetries mentioned above are of this type. On the other hand, the transformation parameters can also depend on space-time, as we saw in the case of QED. Such symmetries are called local or gauge symmetries. As discussed in Ch. 9, such symmetries require the presence of some new particles in the theory, called gauge bosons in general.

13.2 Groups and symmetries

Associated with every symmetry is a mathematical object known as a *symmetry group*. In order to gain an understanding of symmetries, we will need to learn some group theory. Our discussion will by no means be complete or even rigorous. Interested readers should consult the any textbook on group theory and its applications in physics.

13.2.1 Symmetry group

A group is a set of 'elements' in which two elements can be combined by some operation satisfying four properties described below. The operation in general is called *group multiplication*, although it may not have anything to do with ordinary multiplication of numbers. We will denote this operation by a small circle ($\circ$) and the elements of the group by x, y, etc. The operation should have the following properties:

1. The 'product' $x \circ y$ must belong to the group for all elements x, y.

2. There must be an element e in the group such that $e \circ x = x \circ e = x$ for each element x. This e is called the *identity*.

3. Each element x must possess an inverse in the group, i.e., there must exist another element called x^{-1} such that $x \circ x^{-1} = x^{-1} \circ x = e$.

4. The group multiplication must be associative, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$ for all elements x, y, z.

□ **Exercise 13.1** Show that the following set of elements form a group under the specified "group multiplication" operation. Identify the identity element in each case, and the inverse of a typical element.

a) Integers under addition of numbers;

b) Non-zero real numbers under multiplication of numbers;

c) Complex numbers of the form $e^{i\theta}$ under multiplication of complex numbers;

d) n-dimensional square matrices with non-zero determinant under matrix multiplication;

e) n-dimensional unitary matrices under matrix multiplication.

□ **Exercise 13.2** Show that the following set of elements *do not* form a group under the specified "group multiplication" operation.

a) Positive integers under addition of numbers;

b) All integers under multiplication of numbers;

c) All n-dimensional square matrices under matrix multiplication.

□ **Exercise 13.3** Show that the Lorentz transformation matrices satisfying the condition in Eq. (1.28) form a group under matrix multiplication.

The symmetries of a physical system provide useful constraints on the possible forms of its Lagrangian. A symmetry operation is something that leaves the action unchanged. It is not very difficult to see that all symmetry operations form a group. The combination of two such operations will also not change the action and will therefore be a symmetry operation. Secondly, there is an identity, which is the operation of simply doing nothing to the action. The inverse element of any operation is just undoing the operation. And finally, the product of three such operations means performing the operations in a given order. Since each of them leaves the action unchanged, the result cannot depend on how they may be perceived to be combined together. In other words, the action remains unchanged irrespective of how they are combined together. Thus, symmetry operations satisfy all the conditions for a group. This group is called the *symmetry group* of the theory. The symmetry group includes the Poincaré group in a relativistically covariant theory. This invariance leads to the conservation of 4-momentum and angular momentum, as discussed in §2.4. In addition, the symmetry group can include internal symmetries, on which we shall focus in this chapter.

13.2.2 Examples of continuous symmetry groups

As an example, let us go back to the phase freedom of the free Dirac Lagrangian. The symmetry operations are given by

$$\psi(x) \to e^{-iq\theta}\psi(x)\,, \tag{13.1}$$

as mentioned in Eq. (4.91). Thus the elements of the symmetry group in this case are complex numbers of the form $e^{-iq\theta}$, and two such elements combine by the law of multiplication of complex numbers. Alternatively, these elements can be viewed as 1-dimensional unitary matrices, which combine by the law of multiplication of matrices. For this reason, this symmetry group is denoted by U(1) — the letter 'U' for unitary matrices, and the number in the parenthesis for the dimension of the matrices. For the free Lagrangian, this is a global internal symmetry. We can thus say that the free Dirac Lagrangian in invariant under a global U(1) symmetry.

In QED, we still have the same symmetry, except that the parameters θ can depend on space-time. This is usually encoded in the statement that the QED Lagrangian has a local U(1) symmetry. This is also an internal symmetry.

□ **Exercise 13.4** The orthogonal group in two dimensions is denoted by O(2). The transformations are given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\,. \tag{13.2}$$

Show that these transformations are equivalent to U(1) transformations on the object $z = x + iy$.

One can think of larger symmetries as well. For example, consider the free Lagrangian of two Dirac fields:

$$\mathscr{L} = \overline{\psi}_1 \left(i\partial\!\!\!/ - m_1\right)\psi_1 + \overline{\psi}_2 \left(i\partial\!\!\!/ - m_2\right)\psi_2\,. \tag{13.3}$$

Of course this is invariant under *independent* phase rotations of the two fields, i.e., two independent U(1) transformations. This symmetry is called U(1) × U(1). However, in the special case of $m_1 = m_2$, the symmetry is even larger. This is best seen by constructing the object

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,. \tag{13.4}$$

Here, ψ_1 and ψ_2 are independent spinors, meaning that the components of ψ_1 do not mix with those of ψ_2 under Lorentz transformations. However, here we will be considering an *internal* symmetry transformation in which the elements of two different spinors will mix with each other. For this purpose, it is sufficient to suppress the component structure of each spinor and treat each of them as one single object. The object Ψ can then justifiably be called a doublet. Using this doublet, we can cast the Lagrangian of Eq. (13.3) into the form

$$\mathscr{L} = \overline{\Psi}\,(i\partial\!\!\!/ - m)\,\Psi\,, \tag{13.5}$$

where the thing in the parenthesis is to be understood to be a multiple of a 2×2 unit matrix, and $m = m_1 = m_2$.

Now consider the following transformation on Ψ which changes Ψ to Ψ', given by

$$\Psi' = U\Psi\,, \tag{13.6}$$

where U is a 2×2 unitary matrix. This will keep the Lagrangian of Eq. (13.5) invariant. We have already introduced the group of $n \times n$ unitary matrices in Ex. 13.1 (p 285). For a general n, this group is called the U(n) group. The Lagrangian of Eq. (13.5) thus has a U(2) global symmetry.

□ **Exercise 13.5** Argue that the free Lagrangian of n Dirac fields has a U(n) symmetry if the masses of all the fermions are equal.

It is convenient to think of the group U(2) as a product of two subgroups. Any unitary matrix can be written as a product of two objects, a unitary matrix with determinant 1 and an overall phase. Both these parts form a group among themselves. The second of these is the group of phase transformations discussed earlier, which is called U(1). The first group is called SU(2), where the 'S' stands for 'special', indicating the unit determinant. Thus U(2) can be written as SU(2) × U(1), and in general U(n) as SU(n) × U(1).

If we encounter the situation when $m = 0$ in Eq. (13.5), the symmetry is even larger. Because now the U(2) symmetry described above is still there, but in addition one can make the transformations defined by

$$\Psi' = V\gamma_5\Psi\,, \tag{13.7}$$

where V is any 2×2 unitary matrix, independent of the matrix U defined in Eq. (13.6). Thus, the symmetry group now is $\mathrm{U}(2) \times \mathrm{U}(2)$, i.e., $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

Such larger symmetries can exist even in the presence of interactions. As an example, we can consider the original theory of Yukawa, which deals with the interaction of nucleons (i.e., neutrons and protons) with pions. The nucleons can be thought of as a doublet, much like the one introduced in Eq. (13.4):

$$\Psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} , \tag{13.8}$$

where ψ_p and ψ_n denote the proton and the neutron fields respectively. There are three pion fields, with charges $+1$, 0 and -1 in units of proton charge. To include them, consider a triplet of real scalar fields:

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} . \tag{13.9}$$

Now consider the following Lagrangian, which includes the free Lagrangian of these fields, as well as their interactions:

$$\mathscr{L} = \overline{\Psi}\,(i\partial\!\!\!/ - m)\,\Psi + \frac{1}{2}(\partial_\mu \Phi)\cdot(\partial^\mu \Phi) - \frac{1}{2}\mu^2 \Phi\cdot\Phi - \frac{1}{4}\lambda\,(\Phi\cdot\Phi)^2 + ig\overline{\Psi}\gamma_5 \tau_a \Psi \phi_a , \tag{13.10}$$

where τ_a are the usual Pauli matrices. The last term includes interactions like $ig\overline{\psi}_p \gamma_5 \psi_n(\phi_1 + i\phi_2)$. Looking at this term, we see that $(\phi_1 + i\phi_2)$ should annihilate a positively charged particle. We identify this with π^+. The properly normalized pion fields are

$$\pi^+ = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) , \quad \pi^- = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) , \quad \pi^0 = \phi_3 . \tag{13.11}$$

The Lagrangian is invariant under the U(1) symmetry of electromagnetism

$$\psi_p \to e^{-ie\theta}\psi_p , \qquad \pi^\pm \to e^{\mp ie\theta}\pi^\pm . \tag{13.12}$$

In addition, this Lagrangian is invariant under a global U(1) symmetry whose transformations are defined by

$$\Psi \to e^{i\theta}\Psi , \qquad \Phi \to \Phi . \tag{13.13}$$

Using a more compact notation, one can write

$$\chi \to e^{iB\theta}\chi\,, \tag{13.14}$$

where χ is any field, and $B = 1$ for the nucleons and zero for the pions. This B is usually called *baryon number*, and the U(1) symmetry the baryon number symmetry.

In addition to this symmetry, one can check that the Lagrangian of Eq. (13.10) is also invariant under the global SU(2) transformations defined by

$$\begin{aligned}\Psi &\to \exp\left(-i\beta_a\frac{\tau_a}{2}\right)\Psi\,,\\ \Phi &\to \exp\left(-i\beta_a T_a\right)\Phi\,,\end{aligned} \tag{13.15}$$

where the τ_a's are the Pauli matrices, and the matrices T_a are given by

$$T_1 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}\,,\quad T_2 = \begin{pmatrix} 0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{pmatrix}\,,\quad T_3 = \begin{pmatrix} 0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\,. \tag{13.16}$$

This SU(2) symmetry is called *isospin symmetry*. Notice that this symmetry requires that the masses of the neutron and the proton be equal, and the masses of the three pions to be equal as well. This is only roughly true in reality, since the neutron and proton masses differ by only about 0.14%, while the mass of π^0 differs from that of $\pi^\pm$ by about 3.5%. This makes the symmetry an approximate symmetry, which will be discussed in §13.3.

□ **Exercise 13.6** Show that the matrix $\exp\left(-i\beta_a\frac{\tau_a}{2}\right)$, which induces the transformation on the nucleon fields, has determinant equal to unity for any value of the parameters β_a.

□ **Exercise 13.7** Write the infinitesimal forms of the isospin transformations of Eq. (13.15). Show that the Lagrangian of Eq. (13.10) is indeed invariant under these infinitesimal transformations. [Hint: You must disregard all terms of order β^2.]

Construct the current generated by the isospin transformation and verify that they are conserved by using the equations of motion.

13.2.3 Generators of continuous groups

The number of elements in any continuous group is infinite. But they can be represented by a finite number of continuous parameters. For

example, if we consider the U(1) transformation mentioned in Eq. (13.1), the parameter is θ, which can take continuous values. For the SU(2) transformation shown in Eq. (13.15), we have three parameters β_a.

□ **Exercise 13.8** A general $n \times n$ complex matrix has $2n^2$ real parameters. Show that if this matrix is unitary, the number of independent real parameters is n^2. If the determinant is unity, the number is $n^2 - 1$.

For any group G, we can write a general element as $U(\beta_1, \beta_2, \cdots, \beta_r)$, where r is the total number of parameters needed to specify a general group element. One can always choose these parameters in such a way that all of them are zero for the identity element of the group. The generators of the group are then defined by

$$T_a = i \left. \frac{\partial U}{\partial \beta_a} \right|_{\beta=0} . \tag{13.17}$$

This means that the infinitesimal elements of the group can be written as

$$1 - i\beta_a T_a , \tag{13.18}$$

using the summation convention for the index a. For a class of groups called Lie groups which includes the unitary groups that we have been discussing, this implies that the general group element is given by

$$U = \exp\left(-i\beta_a T_a\right) , \tag{13.19}$$

where the β_a's need not be small. Obviously, the choices for T_a are not unique. One can always define some linearly independent combinations of the β_a's to be the independent parameters, in which case the generators will also be linear combinations of the T_a's given above.

□ **Exercise 13.9** Show that the generators span a vector space.

A group is defined by the multiplication of all pairs of elements. Thus in order to reproduce the correct multiplication, the generators T_a must satisfy certain properties. For SU(2), for example, the T_a's do not commute, as seen from Eq. (13.16). So the multiplication of

two group elements will be given by the Baker-Campbell-Hausdorff formula:

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B]_- + \frac{1}{12}\Big([A,[A,B]_-]_- - [B,[A,B]_-]_-\Big) + \cdots \right). \quad (13.20)$$

Therefore it is clear that once the commutator of the generators is known, one can evaluate the multiplication of any pair of group elements. For the SU(2) group, the generators can be chosen in such a way that the commutators are given by

$$[T_a, T_b]_- = i\epsilon_{abc} T_c\,, \quad (13.21)$$

where ϵ_{abc} denotes the completely antisymmetric symbol with $\epsilon_{123} = +1$.

If we want to discuss other groups, e.g., any SU(n) group, most of the previous discussion remains unchanged. Only the number of generators is different from 3. For SU(n) it is $n^2 - 1$, as indicated in Ex. 13.8 (p 290). Otherwise, Eq. (13.19) is still valid, with the implied sum over the index a running from $a = 1$ to $n^2 - 1$. The commutator of the generators also have a more general form than that given in Eq. (13.21), viz.:

$$[T_a, T_b]_- = i f_{abc} T_c\,, \quad (13.22)$$

where f_{abc} are called the *structure constants* of the group. Obviously, $f_{abc} = -f_{bac}$. Moreover, the generators can be chosen in a way that the structure constants are completely antisymmetric in the indices, but the values depend on the specific group under consideration. For the group SU(2), $f_{abc} = \epsilon_{abc}$ as shown in Eq. (13.21). The collection of all commutation relations of the generators is called the *algebra* of the group. If all structure constants vanish for a group, it implies that group multiplication is commutative. In this case, the group is called a commutative group, or more commonly, an *Abelian group*. If group multiplication does not commute in general, at least some structure constants will be non-zero, and the group is non-commutative or *non-Abelian*.

13.2.4 Representations

A representation of a group is a set of linear transformations or matrices $R(x)$ defined for all elements x belonging to the group and having the property

$$R(x)R(y) = R(x \circ y)\,, \tag{13.23}$$

where the left hand side is obtained by usual matrix multiplication. The representation is said to be of the same dimension as these matrices. In a given representation, the generators can be chosen to be matrices of the same dimensionality such that Eq. (13.22) is satisfied.

The group SU(n), for example, forms a subspace of the space of all $n \times n$ matrices. In an abstract way, the group SU(n) can be defined to be any group whose elements have a one to one correspondence with the $n \times n$ matrices of unit determinant in such a way that the group multiplication law is satisfied. However, this definition itself implies that we can have a representation of the abstract group SU(n) in terms of $n \times n$ matrices. This representation is called the *fundamental representation* of the SU(n) group. In this representation, the generators are also $n \times n$ matrices and they satisfy the algebra of the group. Usually, they are chosen to satisfy the relation

$$\text{tr}\ (T_a T_b) = \frac{1}{2}\delta_{ab}\,. \tag{13.24}$$

It can be easily checked that for SU(2), the matrices $T_a = \tau_a/2$ satisfy this condition.

The fundamental representation is by no means the only representation for SU(n). For example, for any group there is always a 1-dimensional representation in which all the elements of the group are represented by the number (or the 1×1 matrix) 1. Obviously this satisfies Eq. (13.23). The generators in this representation are given by $T_a = 0$ for all a. This is called the *singlet representation*.

Another important representation can be defined through the structure constants of the group. To see this, let us define the matrices T_a by the following matrix elements:

$$(T_c)_{ab} = -if_{abc}\,. \tag{13.25}$$

Obviously, the dimensionality of these matrices is equal to the number of generators in the group, i.e., $n^2 - 1$ for SU(n). One can now

check that these matrices indeed satisfy the algebra of the group. The representation so defined is called the *adjoint representation* of the group. There are, of course, many other representations of a group. But in this book, we will need only the ones mentioned above.

Representations determine how fields transform under the symmetry group. For example, in Eq. (13.15), the SU(2) doublet Ψ transforms in the fundamental representation and the triplet Φ transforms in the adjoint representation. In general, the transformation of an n-plet is obtained by using the n-dimensional representation of the group.

□ **Exercise 13.10** Verify the Jacobi identity,

$$[T_a,[T_b,T_c]_-]_- + [T_b,[T_c,T_a]_-]_- + [T_c,[T_a,T_b]_-]_- = 0\,, \quad (13.26)$$

which is satisfied because the multiplication of the generators is associative. Use this to verify that the matrices defined in Eq. (13.25) indeed satisfy the algebra given in Eq. (13.22).

13.3 Approximate symmetries

Symmetries have important consequences on field theories, some of which were mentioned at the beginning of this chapter. For example, if a number of fields transform like a multiplet under a given symmetry, the quanta of those fields should have the same mass, as we showed in our examples.

Sometimes real data seems to show the contrary. For example, in §13.2.3, we introduced the isospin symmetry under which the proton and the neutron form a doublet. This should have implied that the proton and neutron have the same mass, as we saw in Eq. (13.10). But in reality, their masses are not the same.

The contradiction is resolved if we realize that isospin is a symmetry of the Lagrangian written in Eq. (13.10), but that is not the complete Lagrangian to describe the physical properties of proton and neutron. We have left out other interactions of these particles. For example, we know that the proton has an electric charge. Therefore, it should interact with the photon field. This means that if we want to write down a Lagrangian that includes all physical properties of the proton and the neutron, we should at least augment that Lagrangian of Eq. (13.10) by the following term:

$$\mathscr{L}' = -e\,\overline{\psi}_p \Gamma_\mu \psi_p A^\mu\,, \quad (13.27)$$

which is the effective electromagnetic interaction of the proton. This involves the charge form factor as well as the anomalous magnetic moment, as we saw in Eq. (11.85). The neutron, on the other hand, is electrically neutral and therefore does not have any charge form factor. Although there can be other interactions involving the neutron and the proton, Eq. (13.27) is sufficient for the purpose of the present argument. This term in the Lagrangian is *not* invariant under the isospin symmetry introduced in Eq. (13.15). In other words, the isospin symmetry is not a symmetry of the complete Lagrangian that can describe interactions of protons and neutrons. What is the point of talking about it then?

The point is that the coupling g that appears in Eq. (13.10) is much larger than the value of e that appears in Eq. (13.27). So, if we consider a process which gets contributions from both types of coupling, the contribution from the coupling g will be larger in general. In that sense, we can think of the Lagrangian of Eq. (13.27) to be a perturbation on top of that of Eq. (13.10). Isospin symmetry is a symmetry of Eq. (13.10). The term in Eq. (13.27) then breaks this symmetry. However, since the effect of breaking is "small" in the sense described above, isospin symmetry can still be called an "approximate symmetry" of the Lagrangian.

The label "approximate symmetry" can also refer to some discrete symmetries. For example, parity is a symmetry of the QED Lagrangian, as we discussed in Ch. 10. However, if we introduce effects of weak interaction, the Lagrangian contains terms which do not respect parity invariance. But weak interactions, as the name indicates, are weak compared to electromagnetic interactions. So parity violating effects are small compared to parity conserving ones. In this sense, parity is an approximate symmetry. The same is true of charge conjugation and time reversal.

13.4 Spontaneous breaking of symmetries

In the last section, we discussed one reason why the consequences of a symmetry may not be realized in the physical world, viz., that there might be some terms in the Lagrangian which do not obey this symmetry. There is another interesting phenomenon which can lead to similar consequences even though the Lagrangian in fact obeys

the symmetry. This phenomenon is called spontaneous symmetry breaking, which we will discuss in this section with some examples.

13.4.1 Discrete symmetry

Consider a theory with only one real scalar field, governed by the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 . \tag{13.28}$$

Clearly, this is symmetric under the following transformation

$$\phi \rightarrow -\phi , \tag{13.29}$$

which produces a discrete symmetry group. This has some interesting consequences. For example, if we treat ϕ as a quantum field, the symmetry says that processes involving odd number of quanta of this field should not take place. This statement has a loophole which we will now investigate.

Using the usual procedure elaborated in connection with Eq. (3.7), we obtain

$$\mathscr{H} = \frac{1}{2}\left(\Pi^2 + (\boldsymbol{\nabla}\phi)^2 + \mu^2\phi^2\right) + \frac{1}{4}\lambda\phi^4 . \tag{13.30}$$

where Π is the canonical momentum, given by $\Pi = \dot{\phi}$. For the moment, let us forget about quantum physics and think of this as the Hamiltonian of a classical field. We can now ask what is the value of the field ϕ for which the field has minimum energy.

Clearly, the terms involving derivatives of ϕ are non-negative, so the minimum value for them is zero. In other words, in the ground state, the classical field must be a constant over all space-time. Thus, finding the minimum of energy reduces to the problem of finding the minimum of the non-derivative terms, which are given by

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 . \tag{13.31}$$

This function is usually called the *potential* in field theory parlance. Since in Ch. 1, we argued that the concept of potentials is untenable in relativistic field theories, the introduction of this name may cause some concern. However, the potential defined in Eq. (13.31) is *not*

the potential in the sense of non-relativistic mechanics. For example, the gradient of the function in Eq. (13.31) does not give the force. It is a different kind of physical quantity for which the same name has become conventional.

In any case, at the extrema of the function $V(\phi)$, the classical field $\phi_{\rm cl}$ satisfies the condition

$$\phi_{\rm cl}(\mu^2 + \lambda\phi_{\rm cl}^2) = 0\,. \tag{13.32}$$

There are three solutions to this equation. To determine which one really corresponds to the minimum, we need some more information about the parameters involved.

Let us first discuss the parameter λ. Of course λ has to be real because the Hamiltonian has to be hermitian (or real, if we insist on a purely classical vocabulary). For very large values of the field, the potential, and therefore the Hamiltonian, is dominated by the $\lambda\phi_{\rm cl}^4$ term. Thus, if $\lambda < 0$, the value of $V(\phi)$ becomes smaller and smaller (more and more negative) as $\phi_{\rm cl}$ increases, and the classical potential does not have any minimum. This will be a system without a ground state, which indicates an unstable system. In other words, the existence of a ground state demands that

$$\lambda > 0\,. \tag{13.33}$$

Now, what about μ^2? Again, it has to be real. Normally, we would think of μ^2 as mass squared when we quantize. But classically μ^2 can be either positive or negative. If $\mu^2 > 0$, the potential function looks like Fig. 13.1a and $\phi_{\rm cl} = 0$ gives its minimum. This is not the interesting case for us in the present context.

Figure 13.1: The shape of $V(\phi)$ for (a) $\mu^2 > 0$, (b) $\mu^2 < 0$.

If $\mu^2 < 0$, the potential function looks like Fig. 13.1b and the real minima are obtained at

$$\phi_{\rm cl} = \pm\sqrt{\frac{-\mu^2}{\lambda}}\,. \tag{13.34}$$

Now we have two minima, and they are degenerate, as can be seen by plugging these solutions into Eq. (13.31). The classical system is now as likely to be in one of these minima as in the other. If we have a classical system, it is either in the minimum with the positive square root, or in the one with the negative square root.

Let us now turn to quantum field theory. The classical field now should be interpreted as the expectation value of the field ϕ in the ground state, or the vacuum. This is called the *vacuum expectation value* of the field, or VEV for short. Everything that we have said so far about the classical field will now be valid about the VEV of the field ϕ. Thus if $\mu^2 < 0$, the VEV of the field ϕ can be either of the two solutions given in Eq. (13.34). It does not matter which one, but let us call this value v:

$$\langle 0\,|\phi|\,0\rangle \equiv v \neq 0\,. \tag{13.35}$$

But this creates a problem. So far we have expanded the quantum fields in creation and annihilation operators, e.g. in Eq. (3.18). Since the vacuum state is annihilated by the annihilation operators, we also have $\langle 0|\,a^\dagger(p) = 0$ for any p. Therefore, if we use the expansion of Eq. (3.18), the VEV of the field must vanish, contrary to Eq. (13.35). In short, the field ϕ cannot be treated as a quantum field if $\mu^2 < 0$.

There is, of course, a very easy way out of this problem. We can define $\eta(x)$ by the relation

$$\phi(x) = v + \eta(x)\,, \tag{13.36}$$

where this $\eta(x)$ is a quantum field in the sense of Eq. (3.18), i.e., which has zero VEV and therefore can be expanded in terms of creation and annihilation operators. Then,

$$\langle 0\,|\eta|\,0\rangle = 0\,, \tag{13.37}$$

and so Eq. (13.35) is satisfied. The field $\eta(x)$ can therefore be interpreted as the fluctuations around the classical value of the field, which was denoted by v.

However, in this case, in order to obtain the quantum behavior of the field quanta, we should rewrite the Lagrangian of Eq. (13.28) in terms of the field $\eta(x)$. This can be easily done by using the definition of η in Eq. (13.36), and we obtain

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) - \frac{1}{2}(\mu^2 + 3\lambda v^2)\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 \\ &= \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) - \lambda v^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 \,. \end{aligned} \tag{13.38}$$

In writing these forms, we have used the relation $\mu^2 = -\lambda v^2$ which follows from Eq. (13.34), and also omitted a constant term in the Lagrangian as it has no effect on the physics of the system. What we see from the final form is that the quantum of the field η has a mass m_η given by

$$m_\eta^2 = 2\lambda v^2 \,. \tag{13.39}$$

And, because of the presence of the η^3 term, the Lagrangian is not invariant under the operation of a sign change of the quantum field. So we can have processes with an odd number of η quanta in the external lines.

The Lagrangian of Eq. (13.28) had the symmetry. But the values of the parameters may be such that the ground state of the Hamiltonian does not obey this symmetry. Whenever the ground state is not invariant under a symmetry of the Lagrangian, the phenomenon is called *spontaneous symmetry breaking*. The name derives from the fact that in this case, symmetry breaking is not driven by any external agent, but rather the Lagrangian itself leads to a situation where the symmetry is not obeyed in physical processes. In particular, the fluctuations around this ground state will not also manifest the symmetry.

One can argue at this point that the Lagrangian of Eq. (13.38) does not have the symmetry of Eq. (13.29) anyway, so this amounts to explicit symmetry breaking in the Lagrangian written in terms of the quantum field. But there is an important difference between the two cases. To appreciate it, suppose we wanted to write down a Lagrangian in terms of a scalar field η, without paying any attention to the discrete symmetry of Eq. (13.29). We would have obtained a mass term, an η^3 term and an η^4 term in that case. But the coefficients of these terms would have been completely independent of

one another. On the other hand, in the case of spontaneous symmetry breaking, the coupling constants with these three terms are determined by two parameters, λ and v. This means that there is a relation between these parameters, which is the remnant of the original symmetry.

13.4.2 U(1) symmetry

We now discuss the spontaneous breaking of a continuous symmetry, where some new features emerge. For this, we consider a theory involving a complex scalar field, with the Lagrangian given by

$$\mathscr{L} = (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 . \tag{13.40}$$

This Lagrangian has a global U(1) symmetry corresponding to the transformations

$$\phi(x) \to e^{i\theta} \phi(x) . \tag{13.41}$$

As in §13.4.1, we can argue that the vacuum expectation value of the field ϕ should be independent of space-time, so we need to minimize the potential, which in this case is

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2 . \tag{13.42}$$

Spontaneous symmetry breaking can occur if $\mu^2 < 0$, for which the minimum is obtained if

$$|\langle 0 |\phi| 0 \rangle| = \frac{v}{\sqrt{2}} , \tag{13.43}$$

where

$$v = \sqrt{-\mu^2/\lambda} . \tag{13.44}$$

Only the magnitude of the VEV is determined by the minimization condition. The phase can be arbitrary. The minima corresponding to all values of this phase have the same energy, i.e., they are degenerate. The system can be in any of these minima.

Let us assume that the vacuum is where the phase value is zero, i.e., the VEV of ϕ is $v/\sqrt{2}$. Following the same arguments as in

§13.4.1, we can say that $\phi(x)$ cannot be the quantum field in this case. We can write

$$\phi(x) = \frac{1}{\sqrt{2}}\Big(v + \eta(x) + i\zeta(x)\Big)\,, \tag{13.45}$$

where $\eta(x)$ and $\zeta(x)$ have zero VEVs, and can therefore be expanded in creation and annihilation operators. Rewriting the original Lagrangian in terms of these quantum fields, we obtain

$$\mathscr{L} = \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) + \frac{1}{2}(\partial^\mu\zeta)(\partial_\mu\zeta) - \lambda v^2\eta^2 \\ -\lambda v\eta(\eta^2+\zeta^2) - \frac{\lambda}{4}(\eta^2+\zeta^2)^2\,. \tag{13.46}$$

This Lagrangian describes fluctuations around a minimum which is not U(1) symmetric. The symmetry in this case is spontaneously broken. This part is no different from the example in §13.4.1. The field $\eta(x)$ has obtained a mass just as it did there, and the mass is given again by $\sqrt{2\lambda v^2}$.

What seems to be different now is the appearance of a massless field. The field $\zeta(x)$ does not have any mass term in Eq. (13.46). This is a consequence of what is known as *Goldstone's theorem*, whose general form will be discussed in §13.5. But before that, we want to discuss another example of spontaneous symmetry breaking with a larger group.

13.4.3 Non-Abelian symmetry

We now take up the case of an SU(2) symmetric Lagrangian, as in Eq. (13.10). Fields carrying spin cannot obtain any VEV in a Lorentz invariant theory because spin always picks out a preferred direction. So we focus on the terms involving scalars in that Lagrangian, which are:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\Phi)\cdot(\partial^\mu\Phi) - \frac{1}{2}\mu^2\Phi\cdot\Phi - \frac{1}{4}\lambda\,(\Phi\cdot\Phi)^2\,, \tag{13.47}$$

where Φ is a triplet of scalars defined in Eq. (13.9).

Following now the same kind of arguments as in the previous examples, we can conclude that the symmetry will be broken if $\mu^2 < 0$. The VEV of the multiplet Φ will satisfy the relation

$$\langle 0\,|\Phi\cdot\Phi|\,0\rangle = v^2\,, \tag{13.48}$$

where v is given by Eq. (13.44). Just as in the previous example, many vacuum states are possible with this constraint. Let us assume that our system is in the vacuum state where

$$\Phi_{\rm cl} \equiv \langle 0|\Phi|0\rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} . \tag{13.49}$$

One can now check that although the quantum field corresponding to ϕ_3 is massive, the fields ϕ_1 and ϕ_2 are massless. This constitutes another example of Goldstone's theorem, which we now discuss.

□ **Exercise 13.11** Write down the Lagrangian using the quantum fields around the vacuum defined in Eq. (13.49). Show that this Lagrangian still has a U(1) symmetry under which $\phi_+ = (\phi_1 + i\phi_2)/\sqrt{2}$ changes to $e^{i\theta}\phi_+$ and ϕ_3 remains unchanged.

13.5 Goldstone's theorem

13.5.1 Appearance of massless states

In this section we give a model-independent proof of the statement that the spontaneous breaking of a continuous symmetry leads to the existence of a massless scalar. When we say model-independent, we mean that we assume no specific mechanism for symmetry breaking, such as some potential of special form or the non-vanishing VEV of some field. By spontaneous symmetry breaking, we only mean that the vacuum does not have the symmetry of the Lagrangian. We shall stick to internal symmetries only, and shall not deal with the breaking of Poincaré invariance. These invariances have to be maintained by the S-matrix if the fundamental processes obey the postulates of special relativity.

A continuous symmetry implies a conserved current j^μ through Noether's theorem. The usual interpretation of it is the existence of a charge Q, defined in Eq. (2.49), which is time-independent. In the case of spontaneous symmetry breaking however, this interpretation is untenable, because the spatial integral over j^0 may diverge due to poor convergence properties of some field operators at spatial infinity. For example, if we consider the vacuum expectation value of the operator $Q^2(t)$, we obtain

$$\left\langle 0 \left| Q^2(t) \right| 0 \right\rangle = \int d^3x \ \langle 0|j_0(x)Q(t)|0\rangle \ . \tag{13.50}$$

Using the generators $\mathscr{P}$ of translational symmetry which are the momenta, we can write $j_0(x) = e^{i\mathscr{P}\cdot x} j_0(0) e^{-i\mathscr{P}\cdot x}$ etc. Since the vacuum has to be translationally invariant, $e^{-i\mathscr{P}\cdot x}\,|0\rangle = |0\rangle$, so that

$$\left\langle 0 \left| Q^2(t) \right| 0 \right\rangle = \int d^3x \, \left\langle 0 \left| j^0(0) Q(0) \right| 0 \right\rangle . \tag{13.51}$$

The integrand is independent of x since it contains operators at the space-time origin only, and is non-vanishing since

$$Q(0)\,|0\rangle \neq 0 \tag{13.52}$$

when a symmetry is spontaneously broken. Therefore, the integral diverges.

The problem can be reformulated in terms of the field operators. Since Q is the generator of the corresponding symmetry, the vacuum state transforms as

$$|0\rangle \rightarrow |Q\rangle \equiv e^{-i\epsilon Q}\,|0\rangle \tag{13.53}$$

under the symmetry transformation. But $Q\,|0\rangle \neq 0$ as mentioned above. So $|Q\rangle$ is not the vacuum. Since the vacuum was defined as the state annihilated by all annihilation operators, there must be some quantum field A whose expectation value in $|Q\rangle$ is non-vanishing. Using the definition of $|Q\rangle$, we can then write

$$\left\langle 0 \left| e^{i\epsilon Q} A e^{-i\epsilon Q} \right| 0 \right\rangle \neq 0 . \tag{13.54}$$

Since $\langle 0\,|A|\,0\rangle = 0$ for a quantum field, this gives

$$\langle 0\,||Q(t), A]_-|\,0\rangle = a \neq 0 . \tag{13.55}$$

Inserting a complete set of momentum eigenstates as intermediate states and using the translational symmetry as indicated above, we obtain

$$a = \int d^3x \left\langle 0 \left| \left[j^0(x), A \right]_- \right| 0 \right\rangle$$

$$= \sum_n \int d^3x \Big(\langle 0 | j^0(0) | n \rangle \langle n | A | 0 \rangle e^{-ip_n \cdot x} - \langle 0 | A | n \rangle \langle n | j^0(0) | 0 \rangle e^{ip_n \cdot x} \Big)$$

$$= \sum_n (2\pi)^3 \delta^3(\boldsymbol{p}_n) \Big(\langle 0 | j^0(0) | n \rangle \langle n | A | 0 \rangle e^{-iE_n t} - \langle 0 | A | n \rangle \langle n | j^0(0) | 0 \rangle e^{iE_n t} \Big), \quad (13.56)$$

performing the $\boldsymbol{x}$ integration in the last step. On the other hand, we notice that, since $\partial_\mu j^\mu = 0$,

$$\begin{aligned} \frac{da}{dt} &= \int d^3x \left\langle 0 \left| \left[\frac{d}{dt} j^0(x), A \right]_- \right| 0 \right\rangle \\ &= \int d^3x \left\langle 0 \left| [\partial_\mu j^\mu, A]_- - [\boldsymbol{\nabla} \cdot \boldsymbol{j}, A]_- \right| 0 \right\rangle \\ &= - \int d^3x \left\langle 0 \left| [\boldsymbol{\nabla} \cdot \boldsymbol{j}, A]_- \right| 0 \right\rangle . \end{aligned} \quad (13.57)$$

The last term involves an integral over all space and can be written as a surface integral at infinity. The surface integral vanishes because the commutator is between A somewhere interior to the volume and j on the surface, operators separated by very large space-like interval. Therefore, it shows that a must be time-independent.

Looking back at Eq. (13.56) now, we try to argue how the right hand side may be time-independent and non-zero. The positive and negative energy parts each are time dependent, and they cannot cancel in general. The time independence can be obtained if, for a state which satisfies the condition $\langle 0 | A | n \rangle \langle n | j^0(0) | 0 \rangle \neq 0$, we also have $E_n = 0$ for $\boldsymbol{p}_n = 0$. This is therefore a massless state, with the same quantum numbers as j_0 and A. This massless state is called a *Nambu-Goldstone boson* associated with the symmetry breaking. This boson need not be an observable state. We will encounter unphysical states of this kind in §13.6.

13.5.2 Examples of Nambu-Goldstone bosons

We have already encountered some massless scalars which arise when symmetries are spontaneously broken by scalar potentials. We also

mentioned that the occurrences of such particles are examples of Goldstone's theorem. Let us now examine these examples in the light of the general proof of the theorem given above.

In §13.4.1, the symmetry was discrete, and we did not find any Nambu-Goldstone boson. In the other two cases discussed in §13.4, the symmetries were continuous, and we found some massless fields which were Nambu-Goldstone bosons. In §13.4.2, we discussed the spontaneous breaking of a theory with a U(1) symmetry. The Lagrangian was written down in Eq. (13.40). Since the interaction terms contain no derivatives, the conserved charge is given by that obtained from the free Lagrangian, Eq. (3.57). Let us use the field ϕ in terms of its real and imaginary parts (or, in quantum language, its hermitian and anti-hermitian parts) as shown in Eq. (3.46). In terms of these, the charge can be written as

$$Q = q \int d^3x \, \left[\dot{\phi}_1 \phi_2 - \dot{\phi}_2 \phi_1\right] . \tag{13.58}$$

It is now easy to calculate the commutators of this charge with ϕ_1 and ϕ_2, which give

$$\begin{aligned} [Q, \phi_1]_- &= -iq\phi_2 , \\ [Q, \phi_2]_- &= +iq\phi_1 . \end{aligned} \tag{13.59}$$

We chose the VEV of the field ϕ in the direction of its real part, which means that ϕ_2 has a zero VEV, whereas ϕ_1 has a non-zero VEV denoted by v. Thus Eq. (13.59) shows the existence of an operator, viz. ϕ_2, whose commutator with the charge has a non-zero VEV. In this case, ϕ_2 plays the role of the operator A introduced in §13.5.1. According to the general proof, the Nambu-Goldstone boson will be a state $|n\rangle$ such that $\langle 0|\phi_2|n\rangle \neq 0$. This is certainly satisfied if $|n\rangle$ is the quantum of the field ϕ_2 itself. Indeed, that is what we found in §13.4.2, where we had denoted the imaginary part of the field ϕ by ζ.

In the other example of §13.4.3, we started with an SU(2) symmetry, which had three generators. After symmetry breaking, a U(1) group remains unbroken, as indicated in Ex. 13.11 (p 301). This means that the Lagrangian after the symmetry breaking does not change if acted upon by the elements of this group. From the generators given in Eq. (13.16) and the vacuum expectation value in Eq.

(13.49), it is clear that

$$T_3 \Phi_{\rm cl} = 0\,, \tag{13.60}$$

so that

$$\exp\left(-i\beta_3 T_3\right) \Phi_{\rm cl} = \Phi_{\rm cl}\,. \tag{13.61}$$

The intuitive meaning of this statement is the following. The original symmetry was SU(2), which is locally the same as SO(3), the orthogonal group with three variables. Under the SO(3) transformations, the quantity $\Phi \cdot \Phi$ remains invariant. Once we have spontaneous symmetry breaking, this symmetry cannot be manifest. However if we choose the VEV to be along the ϕ_3 direction, we still have a rotational symmetry in the ϕ_1-ϕ_2 plane, which is exactly the remnant U(1) mentioned in Ex. 13.11 (p 301). We then see that two of the three original generators have been broken. And there are also two Goldstone bosons ϕ_1 and ϕ_2.

□ **Exercise 13.12** Consider the Lagrangian of Eq. (13.47) and write the conserved charges of the SU(2) symmetry. Show that, when the symmetry is broken by Eq. (13.49) the commutators of broken charges with Goldstone bosons have non-vanishing VEVs.

□ **Exercise 13.13** Consider an SU(2)-invariant Lagrangian, similar to that in Eq. (13.47), but this time using a doublet of complex scalar fields. If symmetry is spontaneously broken by a VEV of this multiplet, show that no continuous symmetry is left over, and three Goldstone bosons emerge.

13.5.3 Interaction of Goldstone bosons

Let us go back to the simplest example of the occurrence of Goldstone bosons which was presented in §13.4.2. After spontaneous symmetry breaking, the Lagrangian was written in terms of the quantum fields η and ζ and presented in Eq. (13.46). We now want to show how Goldstone bosons interact. For this, let us first take the example of a scattering process

$$\eta(p) + \zeta(k) \to \eta(p') + \zeta(k')\,. \tag{13.62}$$

At the tree level, the diagrams for this process are given in Fig. 13.2. The Feynman amplitude can be calculated from the Lagrangian of Eq. (13.46) to be

$$i\mathscr{M} = -2i\lambda + \frac{i(-2i\lambda v)^2}{(p+k)^2} + \frac{i(-2i\lambda v)^2}{(p-k')^2} + \frac{i(-6i\lambda v)(-2i\lambda v)}{(k'-k)^2 - m_\eta^2}\,. \tag{13.63}$$

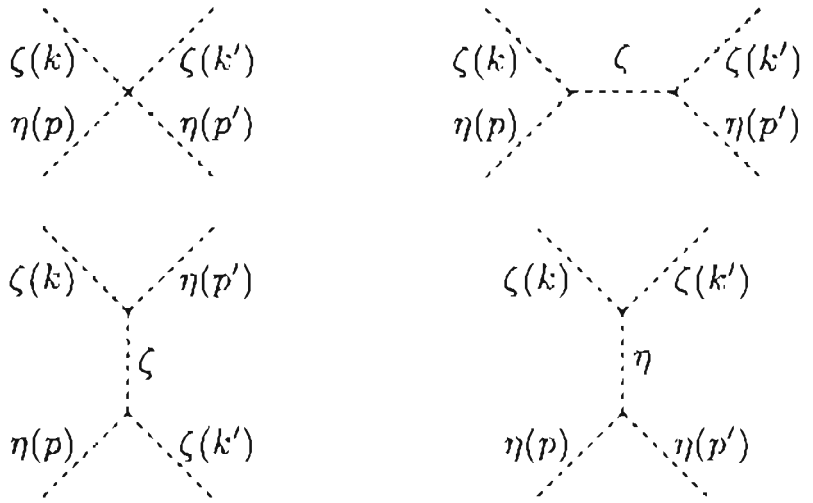


Figure 13.2: Tree-level diagrams for η-ζ elastic scattering.

A comment is in order for the numerical factors on the right hand side. For example, the first term comes from the $\eta^2\zeta^2$ interaction term in Eq. (13.46). The coefficient of this term in the Lagrangian is $-\lambda/2$. However, each of the operators η can either annihilate the initial η particle, or create the final one. This introduces a permutation factor of 2, as mentioned in §6.6. Similarly, we get a factor of 2 for the two factors of ζ. This leads to the numerical co-efficient -2λ of the first term. The coefficients on the other terms appear for similar reasons.

Now, using $k^2 = k'^2 = 0$ and $p^2 = p'^2 = m_\eta^2 = 2\lambda v^2$, we can write it as

$$\begin{aligned} \mathscr{M} &= -2\lambda - \frac{(2\lambda v)^2}{m_\eta^2 + 2p\cdot k} - \frac{(2\lambda v)^2}{m_\eta^2 - 2p\cdot k'} + \frac{(6\lambda v)(2\lambda v)}{m_\eta^2 + 2k\cdot k'} \\ &= -2\lambda\left[1 + \frac{m_\eta^2}{m_\eta^2 + 2p\cdot k} + \frac{m_\eta^2}{m_\eta^2 - 2p\cdot k'} - \frac{3m_\eta^2}{m_\eta^2 - 2k\cdot k'}\right]. \end{aligned} \tag{13.64}$$

Remembering the kinematical relations $p\cdot k = p'\cdot k'$ and $p\cdot k' = p'\cdot k$, we see that this amplitude vanishes if either of the Goldstone bosons has zero 4-momentum, i.e., if either $k^\mu = 0$ or $k'^\mu = 0$.

□ **Exercise 13.14** Consider the ζ-ζ scattering at the tree level in the same model. The initial momenta are k_1, k_2 and the final momenta

k_1', k_2'. Show that the amplitude is given by

$$\mathscr{M} = -2\lambda\left[\frac{k_1\cdot k_2}{k_1\cdot k_2 - \lambda v^2} + \frac{k_1\cdot k_1'}{k_1\cdot k_1' + \lambda v^2} + \frac{k_1\cdot k_2'}{k_1\cdot k_2' + \lambda v^2}\right], \quad (13.65)$$

which vanishes if any one of the Goldstone bosons has zero 4-momentum.

This is a general feature of any interaction of Goldstone bosons. In the limit that the 4-momentum of any external Goldstone boson is zero, the amplitude for any process involving such bosons vanishes. There is a general way to see this directly. Consider, e.g., the model of §13.4.2. We commented that since the VEV of the field ϕ was non-zero, it cannot be treated as a quantum field which can be expanded in creation and annihilation operators. Consequently, we wrote $\phi(x)$ in the form given in Eq. (13.45), where η and ζ have zero VEV so that they can be treated as quantum fields. But this is not the only way that ϕ can be written in terms of quantum fields. Alternatively, we could have written

$$\phi(x) = \frac{v+\eta(x)}{\sqrt{2}} e^{i\zeta(x)/v}. \quad (13.66)$$

If we keep only up to linear terms in the quantum fields, this representation coincides with that of Eq. (13.45). However, this form is illuminating for a different reason. If $\zeta(x)$ were a constant in space-time, it would have been a constant phase for the field ϕ. And the phase of ϕ does not appear in the Lagrangian — that is in fact the U(1) symmetry that the Lagrangian has. Thus, only space-time derivatives of ζ can appear in the Lagrangian. Indeed, substituting Eq. (13.66) in the Lagrangian of Eq. (13.40), we obtain

$$\mathscr{L} = \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) + \frac{1}{2}\left(1+\frac{\eta(x)}{v}\right)^2(\partial^\mu\zeta)(\partial_\mu\zeta) - \frac{1}{2}\mu^2(v+\eta)^2 - \frac{1}{4}\lambda(v+\eta)^4. \quad (13.67)$$

This clearly shows that the field ζ enters into the Lagrangian only through its derivatives. Thus, the Feynman rule for all the vertices involving this field will have at least one power of 4-momentum of Goldstone bosons, and should vanish if the 4-momentum is taken to zero.

□ **Exercise 13.15** Use the Lagrangian of Eq. (13.67) to calculate the Feynman amplitude of the η-ζ elastic scattering process and show that the result coincides with that in Eq. (13.64).

13.6 Higgs mechanism

So far, we have talked about global symmetries only. We have seen in §13.4 that if such symmetries are spontaneously broken, massless bosonic states appear in the theory. If the broken symmetry is local, however, this is not the case. We show this with an example.

We start with a Lagrangian with a local U(1) symmetry:

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2\,, \quad (13.68)$$

where $\mu^2 < 0$. The gauge covariant derivative D_μ is defined by

$$D_\mu = \partial_\mu + ieA_\mu\,. \quad (13.69)$$

The potential of this model is identical with that of the model of §13.4.2 with a global U(1) symmetry. Thus, the solution for the ground state remains the same, and we can define the quantum fields as in Eq. (13.45). When we write down the Lagrangian in terms of the quantum fields, the part coming from the potential is exactly the same as in Eq. (13.46). The interesting part is the modified 'kinetic' term $(D^\mu\phi)^\dagger(D_\mu\phi)$. Using Eq. (13.45), we find

$$\begin{aligned}(D^\mu\phi)^\dagger(D_\mu\phi) = {} & \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) + \frac{1}{2}(\partial^\mu\zeta)(\partial_\mu\zeta) \\ & + ev A^\mu\partial_\mu\zeta + \frac{1}{2}e^2v^2A^\mu A_\mu + \cdots \end{aligned} \quad (13.70)$$

where the dots stand for terms which contain at least three fields and are therefore interaction terms. Comparing with the Proca Lagrangian of Eq. (8.21), we see that the U(1) gauge boson has a mass term after the symmetry is broken, and the mass is given by

$$M_A = ev\,. \quad (13.71)$$

But before we conclude that the gauge boson is massive, there is a problem to settle. There is a term $A^\mu\partial_\mu\zeta$ in Eq. (13.70). Notice that it is quadratic in the fields, so it should be part of the free

Lagrangian of the system. However, it contains two different fields, A^μ and ζ, involving a derivative. If written in terms of creation and annihilation operators, this would imply that we can have Feynman diagrams in which an A^μ changes into a ζ, without any other particle interacting with them. For example, external A^μ can become ζ, and vice versa.

However, this situation can be easily avoided. We know that we have to add a gauge fixing term in order to quantize the theory. Suppose we choose

$$\mathscr{L}_{\rm GF} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu - \xi M_A \zeta\right)^2 . \tag{13.72}$$

The cross term present here would pair up with the problematic term of Eq. (13.70) to make up a total divergence, which can be disregarded in the Lagrangian. Including this gauge fixing term, the terms quadratic in the gauge boson field can be summarized as below:

$$\mathscr{L}_0^{(A)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + \frac{1}{2}M_A^2 A^\mu A_\mu , \tag{13.73}$$

where M_A is given by Eq. (13.71). On the other hand, the quadratic terms involving the ζ-field are given by

$$\mathscr{L}_0^{(\zeta)} = \frac{1}{2}\left[(\partial^\mu\zeta)(\partial_\mu\zeta) - \xi M_A^2\zeta^2\right] . \tag{13.74}$$

From these, we can calculate the propagators of both A and ζ, and find

$$iD_{\mu\nu}(k) = \frac{-i}{k^2 - M_A^2 + i\varepsilon}\left[g_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2 - \xi M_A^2}\right] , \tag{13.75}$$

$$i\Delta(k) = \frac{i}{k^2 - \xi M_A^2} . \tag{13.76}$$

□ **Exercise 13.16** Verify that the propagators of the U(1) gauge field and the field ζ that follow from Eqs. (13.73) and (13.74) are given by Eqs. (13.75) and (13.76).

In quantum field theory, the pole of the propagator gives the mass of the corresponding particle. However, here we seem to have some problem with the poles. Of course the gauge boson propagator has a pole at $k^2 = M_A^2$, which corresponds to a pole at its mass. But it

also has another pole at $k^2 = \xi M_A^2$. Remember that for an unbroken U(1) gauge theory, we argued in §8.4 that ξ is an arbitrary unphysical parameter which was introduced in the gauge fixing term and should not appear in any S-matrix element. The same should be true here. Thus the gauge boson propagator has a pole at an unphysical point as well! And for the ζ-field, this is in fact the only pole.

These are not catastrophes. These facts show that the field ζ is an unphysical field in this theory. The easy way to see that is to use, instead of the representation of the field ϕ given in Eq. (13.45), the alternative one given in Eq. (13.66). In that case, $\zeta(x)/v$ is the phase of the field $\phi(x)$. And any phase, even if it is a space-time dependent one, is irrelevant because of the local U(1) symmetry. More explicitly, the Lagrangian of Eq. (13.68) is invariant under the transformations

$$\begin{aligned} \phi(x) \to \phi'(x) &= e^{-ie\theta(x)}\phi(x)\,, \\ A_\mu(x) \to A'_\mu(x) &= A_\mu(x) + \partial_\mu\theta(x)\,. \end{aligned} \tag{13.77}$$

Therefore, given the representation of the field $\phi(x)$ as in Eq. (13.66), we can always choose a new gauge through Eq. (13.77) without changing any physical implication. In particular, if we choose $\theta(x) = \zeta(x)/ev$, the gauge transformed fields are given by

$$\begin{aligned} \phi'(x) &= \frac{v + \eta(x)}{\sqrt{2}}\,, \\ A'_\mu(x) &= A_\mu(x) + \frac{1}{ev}\partial_\mu\zeta(x)\,. \end{aligned} \tag{13.78}$$

If we substitute these primed fields instead of the unprimed fields, the $\zeta(x)$ field disappears altogether from the Lagrangian. This Lagrangian then contains physical fields only, and the gauge in which this takes place is called the *unitary gauge*. In the language of the general ξ-gauges introduced in Eq. (13.72), this can be identified as making the choice $\xi \to \infty$. In this case, the propagator for the ζ-field has an infinite contribution in the denominator, and therefore any Feynman amplitude involving a ζ-propagator vanishes. For finite values of the gauge parameter ξ, the ζ-field appears in the Lagrangian and therefore in the Feynman rules, but it corresponds to an unphysical degree of freedom nevertheless.

If the symmetry were global, i.e., the gauge field A_μ were not present, the field $\zeta(x)$ would have been the Goldstone boson after

symmetry breaking. This is what we saw in §13.4.2. In the case of local symmetry, we find that not only does the mass term of the field disappear from the Lagrangian, the entire field disappears in the unitary gauge. This is a new feature for local symmetries. Another feature is that the gauge field, which would have been massless if the gauge symmetry were unbroken, develops a mass after symmetry breaking.

The two features are related. In Ch. 8, we remarked that a massless gauge boson has two independent degrees of freedom. In the unbroken Lagrangian of Eq. (13.68), there are thus four independent degrees of freedom in total — two for the gauge field and two for the complex scalar field. In the symmetry broken Lagrangian, the gauge boson is massive. Massive spin-1 particles can have a longitudinal mode of polarization in addition to the two transverse modes. The polarization vector will be explicitly constructed in §15.4. This increase from two to three in the gauge degree of freedom can occur only at the expense of one degree of freedom from the scalar sector. Indeed, $\eta(x)$, being a hermitian field, accounts for only one scalar degree of freedom left over. The other one disappears from the physical degrees of freedom. Metaphorically, one says that the gauge boson has "eaten up" a scalar degree of freedom to get its mass. This phenomenon was noticed by several authors, but is usually referred to as the *Higgs mechanism* after one of them. In Ch. 15, we will see how this phenomenon has played a central role in understanding weak and electromagnetic interactions.

Chapter 14

Yang-Mills theory of non-Abelian gauge fields

In Ch. 9, we constructed QED, which is a gauge theory based on a U(1) symmetry. In this chapter, we will see how to write down the Lagrangians of gauge theories based on non-Abelian symmetries. Such theories are often called Yang-Mills theories after the people who discovered their structure. We will see that they have some novel features which were absent in the Abelian gauge theory of QED.

14.1 Gauge fields of non-Abelian symmetry

Let us recall the crux of the argument of §9.1 where we introduced the U(1) gauge theory. Suppose we have a free Lagrangian with some global symmetries. However, the free Lagrangian is not invariant if these symmetries are considered to be local. If we introduce some spin-1 fields in the theory, we can make the theory locally invariant. These spin-1 fields are the gauge fields.

Let us try to follow the same path for the case of a larger symmetry. We will consider an SU(n) symmetry. So let us start with a Lagrangian which has a global SU(n) symmetry. We encountered such symmetries in §13.2.2. Making a trivial generalization of Eq. (13.5), we can write the free Lagrangian of n Dirac fields in the form

$$\mathscr{L}_\Psi = \overline{\Psi}\,(i\partial\!\!\!/ - m)\,\Psi\,, \tag{14.1}$$

where the masses of all fields are equal. Here Ψ is an n-plet of

Dirac fields which transforms in the fundamental representation of SU(n). This Lagrangian is invariant under global SU(n) transformations which can be written as

$$\Psi \to \Psi' = U\Psi\,, \tag{14.2}$$

where U is any $n \times n$ unitary matrix of unit determinant, and is independent of space-time.

However, if U is a function of space-time, the invariance is lost, because the Lagrangian in terms of Ψ' reads

$$\begin{aligned}\mathscr{L}_{\Psi'} &= \overline{\Psi}'\,(i\partial\!\!\!/ - m)\,\Psi' \\ &= \mathscr{L}_{\Psi} + \overline{\Psi}U^{-1}i\gamma^{\mu}(\partial_{\mu}U)\Psi\,.\end{aligned} \tag{14.3}$$

As in the case of the U(1) gauge theory, we can try to ensure local invariance by introducing new fields into the theory. These fields must be vector fields, as in the case of QED. Since the number of independent parameters in a general SU(n) group element is n^2-1, we need to introduce n^2-1 vector fields to cancel the derivatives of these parameters. Let us denote these vector fields by A^a_μ and write the new Lagrangian in the form

$$\mathscr{L} = \overline{\Psi}\,(iD\!\!\!/ - m)\,\Psi\,, \tag{14.4}$$

where, in analogy with the case of electromagnetism, we define

$$D_\mu = \partial_\mu + igT_a A^a_\mu\,, \tag{14.5}$$

where g is the gauge coupling constant, analogous to the factor e appearing in Eq. (9.15), and T_a are the generators of SU(n) which replace the factor Q appearing in the same equation. The gauge index can be written up or down, whichever is convenient.

If we now make a local transformation defined by

$$\Psi \to \Psi' = U(x)\Psi\,, \qquad A^a_\mu \to A'^a_\mu\,, \tag{14.6}$$

where $U(x)$ are functions on space-time, we can calculate what A'^a_μ should be so that the Lagrangian of Eq. (14.4) remains invariant under this transformation. The transformed Lagrangian is

$$\begin{aligned}\mathscr{L}' &= \overline{\Psi}'\,(i\partial\!\!\!/ - m)\,\Psi' - g\overline{\Psi}'\gamma^{\mu}T_a\Psi' A'^a_\mu \\ &= \overline{\Psi}\,(i\partial\!\!\!/ - m)\,\Psi + \overline{\Psi}U^{-1}i\gamma^{\mu}(\partial_{\mu}U)\Psi - g\overline{\Psi}U^{-1}\gamma^{\mu}T_a U\Psi A'^a_\mu\,.\end{aligned} \tag{14.7}$$

This will be the same Lagrangian as the one given in Eq. (14.4) provided

$$T_a A'^a_\mu = \frac{i}{g}(\partial_\mu U)U^{-1} + UT_a A^a_\mu U^{-1} . \tag{14.8}$$

We have thus gauged the global SU(n) symmetry, and the fields A^a_μ are the gauge fields that have emerged.

□ **Exercise 14.1** Show that, under a local SU(n) transformation, $D_\mu\Psi$ transforms the same way as Ψ, i.e.,

$$(D_\mu\Psi)' = U(x)\,(D_\mu\Psi) . \tag{14.9}$$

□ **Exercise 14.2** Show that, if we started from the Lagrangian of scalar fields in the fundamental representation obeying an SU(n) global symmetry and tried to gauge it, the gauge fields would have transformed as in Eq. (14.8) in that case as well.

Although we started with the definition of the covariant derivative D_μ which mimics the U(1) gauge theory, we notice that some new features have already emerged in the non-Abelian case. To see them, let us write down the transformation properties of the fermion and the gauge fields for infinitesimal transformations. In this case, we can write

$$U(x) \equiv \exp\Big(-ig\beta_a(x)T_a\Big) \approx 1 - ig\beta_a(x)T_a . \tag{14.10}$$

Then for infinitesimal SU(n) transformations,

$$\begin{aligned} \Psi' &= \Psi - ig\beta_a(x)T_a\Psi , \\ A'^a_\mu &= A^a_\mu + \partial_\mu\beta_a + g f_{abc}\beta_b(x)A^c_\mu . \end{aligned} \tag{14.11}$$

In the transformation equation of the gauge fields, the $\partial_\mu\beta_a$ term resembles the U(1) gauge transformation. But the term containing the structure constants is something new. This could not have appeared in a U(1) theory because in an Abelian group, the structure constants vanish.

This is a new feature, but to appreciate its full importance, let us rewrite Eq. (14.11) with all group indices for the case of global transformations, i.e., $\partial_\mu\beta_a = 0$. The transformations of the fermion fields, which are in the fundamental representation, now read

$$\Psi'_i = \Psi_i - ig\beta_b\,[T_b]_{ij}\,\Psi_j . \tag{14.12}$$

For the gauge fields, remembering the definition of the adjoint representation given in Eq. (13.25), we can write

$$A'^a_\mu = A^a_\mu - ig\beta_b \left[T^{(\mathrm{ad})}_b\right]_{ac} A^c_\mu \,, \tag{14.13}$$

where $T^{(\mathrm{ad})}_b$ denotes the generators in the adjoint representation. We conclude that the gauge fields transform in the adjoint representation of the gauge group.

In the case of QED, we found that the photon field remains invariant under a global transformation. This implies that the photon does not carry any electric charge. The non-Abelian gauge fields, on the contrary, transform non-trivially under the gauge group, which means that they carry charges corresponding to the gauge currents. This is the big difference, and it has other implications. We can see them if we try to complete the Lagrangian now by adding the pure gauge Lagrangian, i.e., the part of the Lagrangian which contains the gauge fields only.

14.2 Pure gauge Lagrangian

We have learned that if we start from a globally invariant Lagrangian and try to make the invariance local, we need to replace all partial derivatives by covariant derivatives. We have done it for the fermions. Now we need to do it for the gauge bosons as well.

This is an issue which we overlooked while constructing U(1) gauge theory in Ch. 9, because the consequences are trivial. To confirm that and also to pave the way for non-Abelian theories, it is best to go back to the U(1) theory.

Consider a local U(1) transformation on a fermion field, as given in Eq. (9.4). For infinitesimal U(1) rotations, we can write it as

$$\psi' = \psi - ieQ\theta(x)\psi \,. \tag{14.14}$$

On the other hand, the definition of the covariant derivative for U(1) is given in Eq. (9.15), which is

$$D_\mu\psi = \partial_\mu\psi + ieQA_\mu(x)\psi \,. \tag{14.15}$$

Looking at these two equations, we observe the following. Eq. (14.14) has the original field as the first term, which is independent of the

transformation parameter $\theta(x)$. Similarly, the covariant derivative also contains the partial derivative, which is independent of the gauge fields. As for the other term, we see that if we take the infinitesimal transformation equation and replace $\theta(x)$ by $-A_\mu(x)$, we obtain the corresponding term in the covariant derivative. This is not a coincidence. This is the property which ensures that the effects of the local phase rotation can be compensated by the gauge fields. So the difference between the covariant and the ordinary derivatives for a general field is obtained by taking the infinitesimal change of the multiplet and replacing θ by $-A_\mu$.

If we now try the same thing for the photon fields, we can start from the transformation property of the photon field given in Eq. (9.11), viz.,

$$A'_\nu = A_\nu + \partial_\nu \theta \,. \tag{14.16}$$

Using the same prescription then, we will obtain

$$D_\mu A_\nu = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{14.17}$$

In other words, for the U(1) gauge theory, we could have defined the electromagnetic field tensor as

$$F_{\mu\nu} = D_\mu A_\nu \,, \tag{14.18}$$

and written the pure gauge Lagrangian as

$$-\frac{1}{4}(D_\mu A_\nu)(D^\mu A^\nu) \,. \tag{14.19}$$

This might have been more elegant in some sense, viz., that the Lagrangian of the photon field resembled more closely the gauge-covariant kinetic term for scalar field. But as we see from Eq. (14.17), it would not have made any difference in the content.

Let us now return to non-Abelian gauge theories. Here, the infinitesimal transformation law for a field Ψ in the fundamental representation is given in Eq. (14.11), whereas the covariant derivative appears in Eq. (14.5). Again, we see that the gauge covariant derivative is obtained by replacing the gauge transformation parameters β_a of Eq. (14.11) by $-A^a_\mu$ in the non-trivial term of the covariant derivative. Going back now to the infinitesimal transformation property of

the gauge fields in Eq. (14.11), we can write the covariant derivative of the gauge fields as

$$D_\mu A^a_\nu = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu \equiv F^a_{\mu\nu} . \tag{14.20}$$

Let us now try to see why this definition is useful. If we consider the gauge transformation property of the covariant derivative acting on any field, the transformed quantity does not involve any derivative of the transformation parameters. This was shown for the covariant derivatives of fermion fields in Eq. (9.17) for the U(1) theory and in Eq. (14.9) for an SU(n) theory. We can try the same thing on the covariant derivative of the gauge field. It is convenient to transform Eq. (14.20) to a matrix equation by multiplying both sides by the generators T_a in the fundamental representation and summing over the gauge index a. After making a gauge transformation now, one obtains

$$\begin{aligned}(T_a F^a_{\mu\nu})' &= (D_\mu T_a A^a_\nu)' = \partial_\mu (T_a A^a_\nu)' - \partial_\nu (T_a A^a_\mu)' - g f_{abc} T_a A^{b\prime}_\mu A^{c\prime}_\nu \\ &= \partial_\mu (T_a A^a_\nu)' - \partial_\nu (T_a A^a_\mu)' + ig \left[T_b A^{b\prime}_\mu, T_c A^{c\prime}_\nu \right]_- , \end{aligned} \tag{14.21}$$

using in the last step the definition of the structure constants of the group to write $f_{abc} T_a = -i[T_b, T_c]_-$. If we now use the transformation properties of the gauge fields given in Eq. (14.8), we will obtain

$$(D_\mu T_a A^a_\nu)' = U (D_\mu T_a A^a_\nu) U^{-1} . \tag{14.22}$$

As anticipated, like the covariant derivatives acting on fields in the fundamental representation, here also we do not obtain any term involving $\partial_\mu U$.

□ **Exercise 14.3** Verify Eq. (14.22).

This shows that if we form the following term involving two factors of $F^a_{\mu\nu}$:

$$-\frac{1}{2} \mathrm{tr} \left(T_a F^a_{\mu\nu} T_b F^{\mu\nu}_b \right) , \tag{14.23}$$

it will be gauge invariant and Lorentz invariant. Using $\mathrm{tr}\,(T_a T_b) = \frac{1}{2}\delta_{ab}$ from Eq. (13.24), we finally arrive at the pure Yang-Mills Lagrangian:

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a . \tag{14.24}$$

Again, this looks very much like the U(1) gauge Lagrangian, but it has to be remembered that $F^a_{\mu\nu}$ contains a term which is quadratic in the gauge fields. It is easy to see that a mass term of the form $\frac{1}{2}M^2 A^a_\mu A^\mu_a$ would not respect the gauge symmetry, just as in the U(1) case.

□ **Exercise 14.4** Show that if we take the covariant derivatives in any representation where the generator matrices are denoted by T_a, the field strength tensor can also be defined by the relation

$$[D_\mu, D_\nu]_- = igT_a F^a_{\mu\nu} . \tag{14.25}$$

where D_μ is now considered to be an operator that operates on anything to its right.

□ **Exercise 14.5** Show that the field strength tensors satisfy the identity

$$D_\mu F^a_{\nu\lambda} + D_\nu F^a_{\lambda\mu} + D_\lambda F^a_{\mu\nu} = 0 . \tag{14.26}$$

These are the analogs of the homogeneous Maxwell equations, called the *Bianchi identity*. [Hint: It is easy if you use Eq. (14.25).]

14.3 Interactions of non-Abelian gauge fields

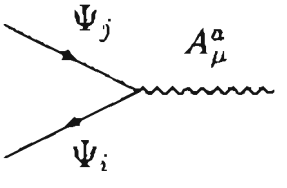
We have encountered two different kinds of interactions of non-Abelian gauge fields so far. First, as in the case of QED, the gauge fields interact with other particles in the theory. And second, because of the presence of the non-linear terms in the definition of $F^a_{\mu\nu}$, the pure gauge Lagrangian involves interaction terms among the gauge fields. Let us discuss these one by one.

14.3.1 Gauge interactions of other particles

The interaction term between fermions and gauge bosons is

$$\mathscr{L}_{\rm int} = -g\overline{\Psi} T_a \gamma^\mu A^a_\mu \Psi . \tag{14.27}$$

The interaction vertex and the corresponding Feynman rule is given by



$$: \; -ig(T_a)_{ij}\gamma_\mu , \tag{14.28}$$

where we have to use the generators of the appropriate representation. For a particle in the singlet representation, $T_a = 0$ for all a, so there is no gauge interaction. And for all particles in non-trivial representations, there will be some gauge interaction.

□ **Exercise 14.6** Consider a gauge theory based on the group SU(2). Instead of the standard hermitian generators, take the generators to be $T_\pm \equiv (T_1 \pm iT_2)/\sqrt{2}$ and $T_0 \equiv T_3$. The gauge fields couple to two fermions which transform like one doublet. Derive the Feynman rules for all gauge interactions of fermions.

For scalars in a gauge theory, the interactions can be similarly calculated. In this case, we should start with the term containing covariant derivatives acting on the scalar fields, i.e.,:

$$(D_\mu\Phi)^\dagger(D^\mu\Phi) = (\partial_\mu\Phi + igT_aA^a_\mu\Phi)^\dagger(\partial^\mu\Phi + igT_aA^\mu_a\Phi)\,. \qquad (14.29)$$

This contains the kinetic terms for the scalar fields in the multiplet Φ, but in addition contains the interaction terms

$$\begin{aligned}\mathscr{L}_{\rm int} &= ig(\partial_\mu\Phi^\dagger)(T_aA^\mu_a\Phi) - ig(T_aA^a_\mu\Phi)^\dagger(\partial^\mu\Phi) \\ &\quad + g^2(T_aA^a_\mu\Phi)^\dagger(T_bA^\mu_b\Phi)\,. \end{aligned} \qquad (14.30)$$

The vertices involve two scalar particles, with either one or two gauge bosons.

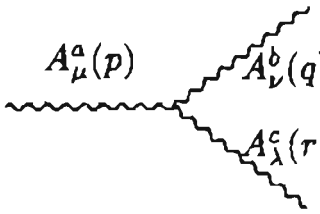
□ **Exercise 14.7** Find the Feynman rules for the interactions of gauge bosons with scalar fields.

14.3.2 Self-interactions of gauge bosons

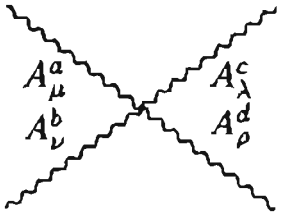
Using the definition of $F^a_{\mu\nu}$ from Eq. (14.20), the pure gauge Lagrangian of Eq. (14.24) can be written as

$$\begin{aligned}\mathscr{L}_{\rm YM} &= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) \\ &\quad + \frac{1}{2}gf_{abc}A^\mu_bA^\nu_c\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \\ &\quad - \frac{1}{4}g^2f_{abc}f_{ade}A^b_\mu A^c_\nu A^\mu_d A^\nu_e\,. \end{aligned} \qquad (14.31)$$

The terms involving only the derivatives of the gauge fields are the kinetic terms, but in addition this Lagrangian contains interactions among the gauge bosons. There are cubic and quartic interactions. The Feynman rules for these vertices are given in Fig. 14.1.



$$: -gf_{abc}\Big[(q-r)_\mu g_{\nu\lambda} + (r-p)_\nu g_{\lambda\mu} + (p-q)_\lambda g_{\mu\nu}\Big]$$



$$: -ig^2\Big[f_{abe}f_{cde}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) + f_{ace}f_{bde}(g_{\mu\rho}g_{\lambda\nu} - g_{\mu\nu}g_{\lambda\rho}) + f_{ade}f_{bce}(g_{\mu\nu}g_{\rho\lambda} - g_{\mu\lambda}g_{\rho\nu})\Big]$$

Figure 14.1: Feynman rules for cubic and quartic interactions among gauge bosons. We have taken the convention that all momenta are coming into the vertices.

As an instructive exercise, let us trace the steps to determining the Feynman rule for the cubic interaction vertex. Obviously, this will come from the term in Eq. (14.31) containing one factor of structure constant. Using the antisymmetry of the structure constants, this term can be rewritten as

$$\mathscr{L}_{\text{cubic}} = g f_{a'b'c'} A^\alpha_{b'} A^\beta_{c'} \left(\partial_\alpha A^{a'}_\beta\right) , \tag{14.32}$$

where we have also changed the notations for the indices appearing in the expression. Let us now refer to the vertex in Fig. 14.1. Suppose all the momenta are pointed towards the vertex. There will be several contributions to this vertex, depending on which particles are annihilated by which of the A^a_μ. There will be thus six terms, like:

$$\begin{aligned}
a'=a,\ b'=b,\ c'=c &\Rightarrow g f_{abc}\, g^\alpha_\nu g^\beta_\lambda\, (-ip_\alpha g_{\mu\beta}) = -igf_{abc}p_\nu g_{\mu\lambda}\,, \\
a'=a,\ b'=c,\ c'=b &\Rightarrow g f_{acb}\, g^\alpha_\lambda g^\beta_\nu\, (-ip_\alpha g_{\mu\beta}) = -igf_{acb}p_\lambda g_{\mu\nu}\,, \\
a'=b,\ b'=a,\ c'=c &\Rightarrow g f_{bac}\, g^\alpha_\mu g^\beta_\lambda\, (-iq_\alpha g_{\nu\beta}) = -igf_{bac}q_\mu g_{\lambda\nu}\,.
\end{aligned} \tag{14.33}$$

Evaluating the other terms similarly and noting the antisymmetry of the structure constants, we finally obtain the Feynman rule for the cubic vertex given in Fig. 14.1. The rule for the quartic vertex can be derived similarly.

□ **Exercise 14.8** Verify the Feynman rule for the quartic vertex given in Fig. 14.1.

14.4 Equations of motion and conserved currents

The gauge invariant kinetic terms of the basic fields of a Yang-Mills theory are already known to us. The Lagrangian is

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \overline{\Psi}i\gamma^\mu D_\mu\Psi + (D^\mu\Phi)^\dagger(D_\mu\Phi) - V(\Psi,\Phi)\,, \quad (14.34)$$

where V includes the mass terms for Ψ and Φ, and may also include interaction terms, but no derivatives. As we mentioned earlier, a mass term like $A^a_\mu A^\mu_a$ would not be gauge invariant.

The Noether currents corresponding to the gauge symmetry can be derived using the definition in Eq. (2.64). We obtain

$$J^\mu_a = \frac{\partial\mathscr{L}}{\partial(\partial_\mu A^b_\nu)}\,\frac{\delta A^b_\nu}{\delta\beta_a} + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\Psi)}\,\frac{\delta\Psi}{\delta\beta_a} + \left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\Phi)}\,\frac{\delta\Phi}{\delta\beta_a} + \text{h.c.}\right], \quad (14.35)$$

where $\delta\Psi$ etc are changes of the fields under infinitesimal gauge transformations characterized by transformation parameters β_a. This gives

$$J^\mu_a = f_{abc}F^{\mu\nu}_b A^c_\nu + \overline{\Psi}\gamma^\mu T_a\Psi + [i(\partial^\mu\Phi)T_a\Phi + \text{h.c.}]\,. \quad (14.36)$$

Notice that the gauge current now includes contributions from the gauge fields. This is because of the fact that the gauge fields transform non-trivially under gauge transformations. As a result, gauge fields carry non-Abelian charge, as was mentioned before.

Using the Lagrangian, we can derive the classical equations of motion for the fields. For the Yang-Mills fields, these are

$$\partial^\mu F^a_{\mu\nu} = J^a_\nu\,. \quad (14.37)$$

This looks similar to the classical equation of motion for the electromagnetic field, Eq. (8.8). The difference, however, is that the currents contain Yang-Mills fields as well. An alternative way of writing Eq. (14.37) is

$$D^\mu F^a_{\mu\nu} = j^a_\nu\,, \quad (14.38)$$

where j^a_ν contains all contributions to the current except those from the gauge fields.

□ **Exercise 14.9** Show the equivalence of Eqs. (14.37) and (14.38).

14.5 Quantization of non-Abelian gauge fields

In Eq. (14.31), we wrote down the Lagrangian of Yang-Mills fields. It contained interaction terms which are new features of Yang-Mills theories. But the quadratic part of the Lagrangian is the same as the Lagrangian for a free electromagnetic field. Thus the problems in quantizing the Lagrangian of gauge fields discussed in §8.2 will appear here as well.

The way to get rid of the problem is same as that described in §8.3, viz., we need to introduce a gauge fixing term. We can choose a covariant gauge fixing term similar to the one for the photon field:

$$\mathscr{L}_{\rm GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu_a)^2 \,. \tag{14.39}$$

Following the same steps as in §8.4, we can now write down the momentum space propagator as

$$D^{ab}_{\mu\nu}(k) = -\frac{\delta_{ab}}{k^2+i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right] . \tag{14.40}$$

This means that in diagrams containing virtual gauge boson lines, we should use the following Feynman rule:

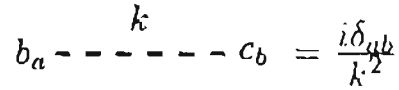
$$A^a_\mu \overset{k}{\sim\!\sim\!\sim\!\sim} A^b_\nu = iD^{ab}_{\mu\nu}(k) \,, \tag{14.41}$$

where $D^{ab}_{\mu\nu}(k)$ is given by Eq. (14.40).

Unlike the electromagnetic case, this is not the end of the story. For reasons which remain outside the scope of the book, gauge invariance cannot be maintained unless one introduces extra unphysical fields in the theory. These are called *Fadeev-Popov ghost fields*, which will be denoted by $c_a(x)$ and *anti-ghost fields* denoted by $b_a(x)$. As the index a implies, there is one such pair corresponding to each gauge boson. These are scalar fields transforming in the adjoint representation, but nevertheless obey anticommutation relations like fermions. For the gauge choice of Eq. (14.39), the Lagrangian involving these fields is given by

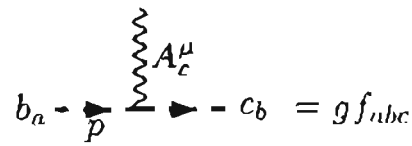
$$\mathscr{L}_{\rm FP} = (\partial^\mu b_a)\left[\partial_\mu c_a + g f_{abc} c_b A^c_\mu\right] . \tag{14.42}$$

The fields b_a and c_a are real in this Lagrangian, so this term is hermitian by itself. They can replace gauge fields in any diagram where there is a closed gauge field loop. As internal lines, their propagators can be read off from Eq. (14.42). This is



$$= \frac{i\delta_{ab}}{k^2}. \tag{14.43}$$

The only interaction of these ghost fields can also be read off from Eq. (14.42). This is an interaction vertex involving a gauge boson. The Feynman rule for this vertex is



$$= g f_{abc} p^\mu. \tag{14.44}$$

This Feynman rule shows why we could avoid discussing the ghost fields for the Abelian case. The structure constants are all zero for an Abelian group, so the ghosts have no interaction with any other field at all. For non-Abelian gauge groups, this is not the case, and one must use these fields in order to obtain correct results. However, it is possible to consistently project out states containing ghost fields from the Hilbert spaces of initial and final states. Then ghost-free physical states evolve only into other ghost-free states. As a result, the ghost fields appear only in loops as virtual particles. For tree level calculations, they are irrelevant.

□ **Exercise 14.10** Consider the 1-loop self-energy diagrams for non-Abelian gauge fields. There will be loops involving other particles that couple with gauge bosons as for QED (fermions, scalars etc.). But now there will be diagrams involving gauge bosons in the loop, as in Fig. 14.2a,b. Calculate these two diagrams in the Feynman-'t Hooft gauge and show that the vacuum polarization is not gauge invariant, i.e., its tensor structure is not $k^2 g_{\mu\nu} - k_\mu k_\nu$, as argued in Ch. 12 for Abelian theories. Now add the contribution of Fig. 14.2c involving a ghost loop and verify that one regains the necessary tensor structure. [Hint: Use dimensional regularization.]

14.6 Quantum Chromodynamics

Non-Abelian gauge theories are important because according to our present level of understanding, all fundamental interactions except

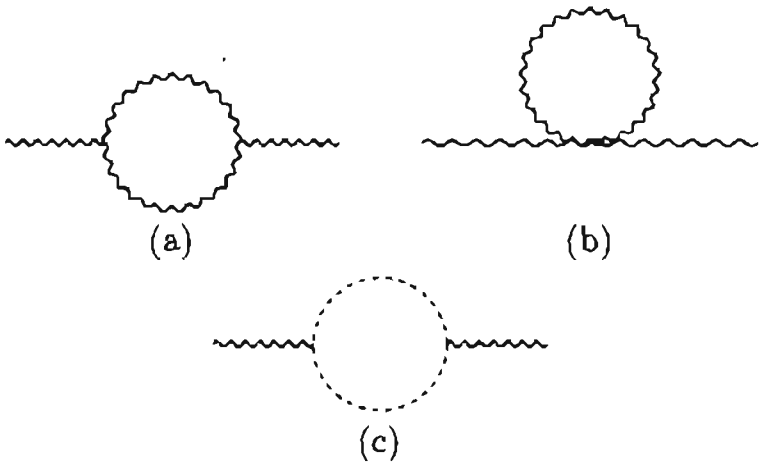


Figure 14.2: Self-energy diagrams for gauge fields in a pure non-Abelian gauge theory.

gravity are described by such theories. The gauge bosons of various non-Abelian symmetries mediate different interactions. Even electromagnetic interactions form part of a bigger non-Abelian gauge theory. The detailed nature of these symmetries for weak and electromagnetic interactions will be discussed in Ch. 15. Here, we briefly look at the strong interactions, which is described by a SU(3) gauge theory.

All known fundamental fermion fields are divided into two broad classes. One class, called *leptons*, such as the electron, muon and neutrinos, do not have any strong interactions. In gauge theory parlance, we can say that these are singlets under the strong interaction gauge group. The other class, called *quarks*, can be described by SU(3) multiplets, and therefore have strong interactions. They are the constituents of strongly interacting particles like the proton and the neutron.

Quarks come in three 'colors'. Denoting these colors by an index a, we can write the free Lagrangian involving the quarks as

$$\mathscr{L} = \sum_A \overline{q}_{Aa} \left(i\partial\!\!\!/ - m_{q_A} \right) q_{Aa} \,, \tag{14.45}$$

where the sum over A runs over different quarks. Obviously this free Lagrangian is invariant under a global SU(3) symmetry under which three colors transform like a triplet. If we try to gauge this color

symmetry, we obtain a theory called *Quantum Chromodynamics*, or QCD for short. This theory will contain eight gauge fields G^a_μ, which are called *gluons*.

Chapter 15

Standard electroweak theory

To the best of our knowledge, there are four different kinds of interactions — strong, electromagnetic, weak and gravitational. Description of gravitational interactions requires general co-ordinate invariance in curved spaces, as in Einstein's general theory of relativity. We have been discussing field theories based on Lorentz invariance, which are inadequate for describing gravitational interactions. There have been attempts to construct quantum theories of gravity using general co-ordinate invariance as the gauge symmetry, but so far all such attempts have led to non-renormalizable theories. Among the other three, electromagnetic interactions are described by the gauge theory of QED, which was discussed in detail in Ch. 9. Strong interactions are described by the non-Abelian gauge theory of QCD, as we mentioned in Ch. 14. In this chapter, we show how weak interactions are also described by a gauge theory. The gauge group required for this theory also includes the gauge group of QED, so this theory is called electroweak theory.

15.1 Gauge group

15.1.1 Choice of gauge group

The earliest known example of a weak interaction process is that of beta decay: $n \to p + e^- + \widehat{\nu}_e$. Among the four particles appearing in the initial and final states of this process, the neutron and the proton are hadrons, i.e., they have strong interactions as well. In analogy with QED, let us try to think of this process as being mediated by

a gauge boson. Suppose we try to keep the neutron and proton in the same multiplet, as we did for the isospin symmetry in Eq. (13.8). Then there will be a vertex of neutron and proton with a gauge boson. The other end of the gauge boson line has to end on a vertex involving the electron and the neutrino, which therefore should also belong to another multiplet of the same group. The resulting diagram would look like Fig. 15.1.

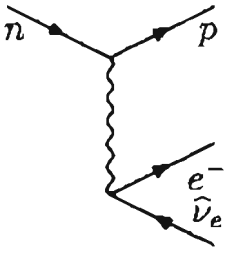


Figure 15.1: β-decay mediated by a gauge boson.

The simplest possibility is that the members of a pair are parts of a doublet. Among the unitary groups, only SU(2) has a doublet representation, and so it indicates that the gauge group to describe weak interactions should either be SU(2) itself, or should contain SU(2) as a subgroup.

Let us consider the first alternative first, since it is more economical than the other. An SU(2) group has three generators. Corresponding to them, there would be three gauge bosons. The interesting point to note is that in the beta-decay process, the electric charge in the hadronic sector changes by one unit. This indicates that there must be gauge bosons which carry one unit of electric charge. These are termed W^+ and its antiparticle W^-. If neutron and proton are taken as the two components of a doublet in an SU(2) gauge theory and one writes down the gauge interactions of this doublet, one finds that the third gauge boson must be uncharged, and has to couple to the neutron at the tree level. Similarly the gauge boson must also couple to the neutrino, which appears in a doublet along with the electron. The only uncharged spin-1 particle which was known at the time when these theories were being formulated was the photon. But the photon cannot couple to uncharged particles.

□ **Exercise 15.1** Consider an SU(2) gauge theory. For a fermion

ψ whose T_3-eigenvalue is t_3 under the gauge group, show that the interactions with the neutral gauge boson W_3^μ is given by

$$t_3 \overline{\psi} \gamma_\mu \psi W_3^\mu \,. \tag{15.1}$$

One can consider various possibilities at this point. One is to put either the electron or the neutrino in the same multiplet as the neutron. The diagram for β-decay would be different in this case. Another alternative is to enlarge the fermion spectrum, including new fermions, such that the neutron and the proton are parts of an SU(2) triplet, with the neutron transforming as the zeroth component. According to Eq. (15.1), the neutron will not have fundamental couplings with the uncharged gauge boson in this case, and the gauge boson can then be identified with the photon. These possibilities are not phenomenologically successful.

Alternatively, we could keep the fermions in doublets, but acknowledge that the neutral gauge boson of the SU(2) we have been considering is not the photon. In this case, if we want to construct a theory that includes the photon as well, we need a larger gauge group. The smallest gauge group which contains one more gauge boson apart from the SU(2) ones is SU(2) $\times$ U(1), or equivalently U(2). This leads to the standard electroweak theory, which has been tremendously successful in describing weak and electromagnetic interactions. This is the theory we want to outline in this chapter.

15.1.2 Pure gauge Lagrangian

The gauge group, as we said, is SU(2) $\times$ U(1). SU(2) has three generators, and U(1) has one. In the doublet representation of SU(2), we can write the generators as $\tau/2$, where the τ's are the Pauli matrices. It is convenient to consider these matrices in the following combinations:

$$\begin{aligned}
\tau_+ &\equiv \frac{1}{\sqrt{2}}(\tau_x + i\tau_y) = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} , \\
\tau_- &\equiv \frac{1}{\sqrt{2}}(\tau_x - i\tau_y) = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} , \\
\tau_3 &\equiv \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\end{aligned} \tag{15.2}$$

The only non-vanishing structure constants corresponding to this choice are

$$\begin{aligned} f_{+-3} &= -f_{-+3} = -i\,, \\ f_{+3+} &= -f_{3++} = i\,, \\ f_{-3-} &= -f_{3--} = -i\,. \end{aligned} \tag{15.3}$$

Note that the structure constants are not completely antisymmetric in this basis. The U(1) generator will have to commute with all SU(2) generators, and therefore must be a multiple of the unit matrix.

Corresponding to the four generators, there are four gauge bosons in this theory. We will call the SU(2) gauge bosons W_μ^+, W_μ^- and W_μ^3. The U(1) gauge boson will be called B_μ. The generator of this U(1) is not the electric charge (it is something called *hypercharge*) and so B^μ is not the photon field. The photon will be identified later. The pure gauge Lagrangian, containing the gauge fields only, can be written as

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{4} W^a_{\mu\nu} W_a^{\dagger\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}\,, \tag{15.4}$$

where

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g f_{abc} W^b_\mu W^c_\nu \tag{15.5}$$

as defined in Eq. (14.20) for general Yang-Mills fields. We have put a dagger in one of the factors of $W^a_{\mu\nu}$ in Eq. (15.4). This is because the index a, b, c take the values $+, -, 3$, and f_{abc} are not real in this basis. For the U(1) part of the theory, the definition of $B_{\mu\nu}$ is similar to that of the field tensor for electromagnetism:

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \tag{15.6}$$

As was observed for a general Yang-Mills theory and for QED, the pure gauge Lagrangian cannot contain any mass term for the gauge bosons since such a term would not be gauge invariant. However, experiments show that the gauge bosons mediating weak interactions are massive.

15.2 Spontaneous symmetry breaking

15.2.1 Introducing the Higgs boson multiplet

The way to solve this problem was indicated in §13.6, where we showed how spontaneous symmetry breaking in a gauge theory leads to the generation of mass for gauge bosons. To implement the same idea in the present context, we need some scalar fields in the theory. Let us introduce a scalar multiplet which is a doublet under the SU(2) part of the gauge group, and write this as

$$\phi \equiv \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} \; : \; (2, \tfrac{1}{2}) \, . \tag{15.7}$$

On the right, we have shown the gauge transformation properties of this multiplet. The '2' indicates that it is a doublet of SU(2), and we have normalized the U(1) charge such that its value is $\frac{1}{2}$ for the multiplet ϕ.

The Lagrangian now contains terms involving ϕ as well. These are

$$\mathscr{L}_\phi = (D^\mu \phi)^\dagger (D_\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \, . \tag{15.8}$$

Here D_μ, as before, is the gauge covariant derivative. Since the gauge group now contains two factors, we should have the gauge bosons of both factors in the definition of D_μ, and these two sets can come with two different coupling constants. In general, when a multiplet transforms like an n-dimensional representation of SU(2) and has a U(1) quantum number Y, we should write

$$D_\mu = \partial_\mu + i g T_a^{(n)} W_\mu^a + i g' Y B_\mu \, , \tag{15.9}$$

where $T_a^{(n)}$ denote the generators of SU(2) in the n-dimensional representation, and Y is the identity matrix times the hypercharge. For the doublet ϕ, the SU(2) generators are $\tau_a/2$, and we can write

$$\begin{aligned} D_\mu \phi &= (\partial_\mu + i g \frac{\tau_a}{2} W_\mu^a + i \frac{g'}{2} B_\mu) \phi \\ &= \partial_\mu \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} g W_\mu^3 + g' B_\mu & \sqrt{2} g W_\mu^+ \\ \sqrt{2} g W_\mu^- & -g W_\mu^3 + g' B_\mu \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} . \end{aligned} \tag{15.10}$$

Let us now suppose, as in various examples on spontaneous symmetry breaking in Ch. 13, that $\mu^2 < 0$ in Eq. (15.8). In this case, the gauge symmetry will be spontaneously broken. The minimum for the scalar potential will be obtained for scalar field configurations given by

$$\phi_{\rm cl} = \frac{v}{\sqrt{2}}\chi\,, \tag{15.11}$$

where χ is a doublet satisfying $\chi^\dagger\chi = 1$, and

$$v = \sqrt{-\mu^2/\lambda}\,. \tag{15.12}$$

Without any loss of generality, we will consider our system to be in the vacuum state

$$\phi_{\rm cl} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}\,, \tag{15.13}$$

i.e., $\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Any other vacuum satisfying Eq. (15.11) can be reached from this by a global SU(2) × U(1) transformation. In this vacuum state, the generators T_1 and T_2 are no more part of the symmetry, since $\tau_x\phi_{\rm cl} \neq 0$, $\tau_y\phi_{\rm cl} \neq 0$. The diagonal generators T_3 and the U(1) generator Y also do not annihilate the vacuum state. However, notice that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,. \tag{15.14}$$

Thus there is one diagonal generator which annihilates the vacuum. This is a linear combination of T_3 and Y, given by

$$Q = T_3 + Y\,. \tag{15.15}$$

The original gauge symmetry is therefore broken down to a U(1) symmetry generated by Q. This is not the original U(1) part of the symmetry group. As we have shown, its generator is actually a combination of an SU(2) generator and the original U(1) generator. To make this distinction, we will write the original symmetry from now on as SU(2) × $\mathrm{U(1)_Y}$, whereas the remnant symmetry will be called $\mathrm{U(1)_Q}$. The latter is in fact the electromagnetic gauge symmetry,

as we will see from the couplings of its gauge boson with fermions. For now, notice that the superscripts on the components of ϕ which appeared in Eq. (15.7) are indeed the electric charges of the corresponding fields.

15.2.2 Gauge boson masses

Since ϕ_0 cannot be described by creation and annihilation operators, we can use the technique employed in Ch. 13 and write

$$\phi(x) = \begin{pmatrix} \phi^+(x) \\ \dfrac{v + H(x) + i\zeta(x)}{\sqrt{2}} \end{pmatrix} , \tag{15.16}$$

where ϕ^+, H and ζ are all quantum fields with vanishing expectation values in the vacuum state. Using this decomposition, we can write the Lagrangian terms of Eq. (15.8) in terms of the quantum fields. Using the expression for $D_\mu\phi$ from Eq. (15.10) and substituting Eq. (15.16) in it, we find various kinds of terms. One of these pick up only the VEV part of the scalar field ϕ. These quadratic terms in the gauge fields are

$$\mathscr{L}_2 = \left\| \frac{i}{2} \begin{pmatrix} gW_\mu^3 + g'B_\mu & \sqrt{2}gW_\mu^+ \\ \sqrt{2}gW_\mu^- & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{v}{\sqrt{2}} \end{pmatrix} \right\|^2 , \tag{15.17}$$

where $\|\chi\|^2 = \chi^\dagger\chi$ for any column matrix χ. Explicitly, this gives

$$\mathscr{L}_2 = \frac{1}{4}g^2v^2W_\mu^+W_\mu^- + \frac{1}{8}v^2(-gW_\mu^3 + g'B_\mu)^2 . \tag{15.18}$$

For any charged spin-1 field V_μ of mass M, the mass term in the Lagrangian is $M^2V_\mu^\dagger V^\mu$. Thus, the mass of the charged W-boson is

$$M_W = \frac{1}{2}gv . \tag{15.19}$$

As for the combination $-gW_\mu^3 + g'B_\mu$ of neutral gauge bosons appearing in Eq. (15.18), we first define the combinations

$$\begin{aligned} Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu , \\ A_\mu &= \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu . \end{aligned} \tag{15.20}$$

where θ_W is called the *Weinberg angle*, defined by

$$\tan\theta_W = \frac{g'}{g}\,. \tag{15.21}$$

Then the kinetic terms for Z_μ and A_μ are properly normalized because this is an orthogonal transformation, and the last term of Eq. (15.18) can be written as $\frac{1}{8}v^2(g^2+g'^2)Z_\mu Z^\mu$. Since the mass term for a real spin-1 field V^μ is $\frac{1}{2}M^2V_\mu V^\mu$, we conclude that the mass of the Z particle is

$$M_Z = \frac{1}{2}\left(g^2+g'^2\right)^{1/2}v = \frac{M_W}{\cos\theta_W}\,. \tag{15.22}$$

We see that A^μ remains massless even after the symmetry breaking process. As we mentioned, only a part of the gauge symmetry is broken. There is a remaining U(1) symmetry. A_μ is the gauge boson corresponding to this symmetry. In §15.3, we will see that this gauge boson is in fact the photon, and so the remnant symmetry is the U(1) symmetry of QED.

15.2.3 Scalar modes

In §13.6, we discussed that a gauge boson can become massive in a spontaneously broken theory by eating up a scalar degree of freedom. This scalar disappears altogether from the theory in the unitary gauge. In other gauges, it appears with a propagator containing an unphysical pole and therefore represents an unphysical mode. In the electroweak theory so far, we have seen that three gauge bosons, the W^+, W^- and the Z, become massive. There must be three unphysical modes corresponding to them.

One way to identify these modes is to look at the ϕ-Lagrangian in Eq. (15.8) and substitute the expression for ϕ in terms of the quantum fields which appears in Eq. (15.16). Among other things, we will find terms like

$$iM_W(W^+_\mu\partial^\mu\phi^- - W^-_\mu\partial^\mu\phi^+) - M_Z Z_\mu\partial^\mu\zeta\,. \tag{15.23}$$

As argued in §13.6, this shows that the W^+ can be annihilated at a point with the creation of a ϕ^+, without any other particle interacting with them. This shows that ϕ^+ is indeed the unphysical mode eaten up by the W in the process of symmetry breaking. Similarly, W^-

eats up ϕ^-, and Z eats up the field ζ. We can see this clearly if we go to the unitary gauge by writing the doublet ϕ in a manner similar to what was done in Eq. (13.66). In such a gauge, only the observable degrees of freedom appear in the Lagrangian, and the particle interpretation is straight forward.

□ **Exercise 15.2** Show that Eq. (15.23) contains all terms in Eq. (15.8) which involve a single derivative.

□ **Exercise 15.3** ⋆ Construct the unitary gauge explicitly by writing the doublet ϕ in terms of quantum fields so that after a gauge transformation the unphysical degrees of freedom disappear from the Lagrangian.

For calculational purposes, however, it is more convenient to use a gauge fixing term such that the mixed quadratic terms of Eq. (15.23) disappear from the Lagrangian. As we discussed in §13.6, this can be achieved if we choose the gauge fixing term to be

$$\mathscr{L}_{\rm GF} = -\frac{1}{\xi}\left|\partial_\mu W^+_\mu + i\xi M_W \phi^+\right|^2 - \frac{1}{2\xi}\left(\partial_\mu Z^\mu + \xi M_Z \zeta\right)^2 . \quad (15.24)$$

The terms of Eq. (15.23) now pair with the cross terms from this equation to become total divergence terms, and we are left with the gauge boson propagators in the form

$$iD^{(W)}_{\mu\nu}(k) = \frac{-i}{k^2 - M_W^2 + i\varepsilon}\left[g_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2 - \xi M_W^2}\right] , \quad (15.25)$$

$$iD^{(Z)}_{\mu\nu}(k) = \frac{-i}{k^2 - M_Z^2 + i\varepsilon}\left[g_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2 - \xi M_Z^2}\right] . \quad (15.26)$$

The unphysical scalar propagators are similar to that obtained in Eq. (13.76):

$$i\Delta^{(\phi^+)}(k) = \frac{i}{k^2 - \xi M_W^2} , \quad (15.27)$$

$$i\Delta^{(\zeta)}(k) = \frac{i}{k^2 - \xi M_Z^2} . \quad (15.28)$$

This choice of gauge is called the R_ξ gauge. The subscript ξ in the name refers to the gauge parameter, and the letter R indicates that the power-counting renormalizability of the theory is obvious in this gauge since all propagators scale like $1/k^2$ for large values of k.

There is also a physical scalar mode called the Higgs boson, which corresponds to the quantum field H in Eq. (15.16). It is an uncharged scalar boson with mass given by

$$M_H^2 = 2\lambda v^2 . \tag{15.29}$$

Existence of this Higgs boson can be counted among the definitive predictions of the standard model.

15.3 Fermions in the theory

When the electroweak theory was first formulated, it appeared only as a theory of leptons. Although subsequently the quarks were included in the framework and it is now known that the theory suffers from some inconsistencies without the quarks, it is simplest to discuss first the leptonic sector of the theory without bringing the quarks in. Moreover, we will start with only the electron and the electron-neutrino, leaving the other leptons and quarks for §15.3.4.

15.3.1 Gauge interactions

The electron and the electron-neutrino form what is called the first generation of leptons,. Parity is known to be violated in weak interactions, so it is expected that the different chiral components will have different interactions. In the standard electroweak model, these fermions are supposed to transform like the following gauge multiplets:

$$\Psi_{eL} \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} : (2, -\frac{1}{2}) ,$$
$$e_R : (1, -1) . \tag{15.30}$$

It should be noted that we have written down both the left and right chiral projections of the electron, but only the left chiral projection for the neutrino. At the time of the formulation of the theory and for quite a while afterwards, all laboratory experiments showed that neutrinos always appeared to be left handed, and that is why the right chiral neutrino state was assumed not to exist in the standard model. As we will see, a consequence of this assumption is that the neutrinos turn out to be massless. In more recent times there

has been some indication that neutrinos may have mass. If these observations are confirmed, some modification of the standard model would be necessary. It is not clear what the correct modification would be. However, neutrino masses, even if non-zero, must be small. For most processes which do not depend crucially on a non-zero neutrino mass, one can still obtain a good estimate by taking the neutrinos to be massless. For this reason, we present here the original version of the standard model, without any possible modifications which might be prompted by non-zero neutrino masses.

We can write down the gauge interactions for the leptons. The gauge covariant kinetic terms for the leptons are

$$\mathscr{L} = \overline{\Psi}_{eL} i\gamma^\mu D_\mu \Psi_{eL} + \bar{e}_R i\gamma^\mu D_\mu e_R \,. \tag{15.31}$$

The general form for D_μ was given in Eq. (15.9). Since e_R is an SU(2) singlet, the SU(2) generators $T_a = 0$ for it. Using $Q = T_3 + Y$, we see that both e_L and e_R have electric charge -1, while ν_{eL} is uncharged. This justifies the hypercharge assignments for these fields.

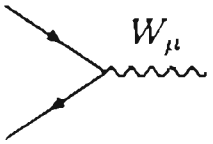
□ **Exercise 15.4** Show that the free Lagrangian of a Dirac field can be written in terms of the chiral projections as

$$\mathscr{L} = \overline{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \overline{\psi}_R i\gamma^\mu \partial_\mu \psi_R - m(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L) \,. \tag{15.32}$$

The derivative terms in Eq. (15.31) give the usual kinetic terms for the electron and the neutrino fields. The other terms in Eq. (15.31) are the interaction terms of the leptons with the gauge bosons:

$$\begin{aligned}
\mathscr{L}_{\text{int}} &= -\overline{\Psi}_{eL}\gamma^\mu \left(g\frac{\tau^a}{2} W_\mu^a - \frac{1}{2} g' B_\mu \right) \Psi_{eL} + \bar{e}_R \gamma^\mu g' B_\mu e_R \\
&= -\frac{g}{\sqrt{2}} \left(\bar{\nu}_{eL}\gamma^\mu e_L W_\mu^+ + \bar{e}_L \gamma^\mu \nu_{eL} W_\mu^- \right) \\
&\quad -\frac{1}{2}\bar{\nu}_{eL}\gamma^\mu \nu_{eL}(gW_\mu^3 - g'B_\mu) \\
&\quad +\frac{1}{2}\bar{e}_L\gamma^\mu e_L(gW_\mu^3 + g'B_\mu) + \bar{e}_R\gamma^\mu g' B_\mu e_R \,.
\end{aligned} \tag{15.33}$$

The interactions involving $W^\pm$ are called *charged current interactions* because these gauge bosons are electrically charged. These lead to the following Feynman rules:



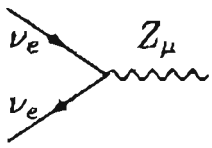
$$: \quad -\frac{ig}{\sqrt{2}}\gamma^\mu \mathsf{L} \,, \tag{15.34}$$

where $\mathsf{L} = \frac{1}{2}(1-\gamma_5)$, the projection operator for left chirality defined in §4.7. The incoming fermion line is ν_e and the outgoing one electron if the gauge boson going out is W^+, and the converse if the gauge boson going out is W^-.

Let us now look at the interactions of neutral gauge bosons in Eq. (15.33). These are called *neutral current interactions*, and we should rewrite them using the neutral boson eigenstates Z_μ and the photon. Looking back at the definition of the Z boson in Eq. (15.20), we see that neutrinos have neutral current interactions with only the Z boson. This should have been expected. Neutrinos do not have electric charge, and therefore cannot have any tree-level interaction with the photon. The neutral current interaction of neutrinos is given by

$$\begin{aligned}\mathscr{L}_{\rm nc}^{(\nu_e)} &= -\frac{1}{2}\sqrt{g^2+g'^2}\;\bar{\nu}_{eL}\gamma^\mu\nu_{eL}Z_\mu \\ &= -\frac{g}{2\cos\theta_W}\,\bar{\nu}_e\gamma^\mu\mathsf{L}\nu_e Z_\mu\,. \end{aligned} \tag{15.35}$$

The Feynman rule for the neutrino vertex with the Z-boson is thus given by



$$: \quad -\frac{ig}{2\cos\theta_W}\,\gamma^\mu\mathsf{L}\,. \tag{15.36}$$

As for the electron, we can rewrite the terms containing the electron field in Eq. (15.33) as

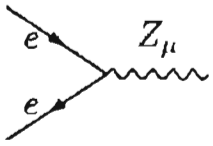
$$\mathscr{L}_{\rm nc}^{(e)} = \frac{1}{2}\bar{e}\gamma^\mu\mathsf{L}e(gW_\mu^3 - g'B_\mu) + g'\bar{e}\gamma^\mu eB_\mu\,, \tag{15.37}$$

using $\mathsf{L}+\mathsf{R}=1$. In terms of the gauge boson eigenstates, this can be written as

$$\begin{aligned}\mathscr{L}_{\rm nc}^{(e)} &= \frac{g}{2\cos\theta_W}\bar{e}\gamma^\mu\mathsf{L}eZ_\mu + g'\bar{e}\gamma^\mu e(-\sin\theta_W Z_\mu + \cos\theta_W A_\mu) \\ &= \frac{g}{2\cos\theta_W}\bar{e}\gamma^\mu\Big[\mathsf{L} - 2\sin^2\theta_W\Big]eZ_\mu + g'\cos\theta_W\bar{e}\gamma^\mu eA_\mu\,, \end{aligned} \tag{15.38}$$

using the definition of the Weinberg angle from Eq. (15.21). The

Feynman rule for the Z-interaction of the electron is thus given by



$$: \quad \frac{ig}{2\cos\theta_W}\,\gamma^\mu(\mathsf{L} - 2\sin^2\theta_W)\,. \tag{15.39}$$

The interaction between the electron and A_μ agrees with QED if we identify the magnitude of the electron charge as

$$e = g'\cos\theta_W = g\sin\theta_W\,. \tag{15.40}$$

❐ **Exercise 15.5** Show that, for any fermion of charge eQ whose left-chiral component has a T_3-eigenvalue of t_3 and the right-chiral component is an SU(2) singlet, the Feynman rule for the Z-interaction is

$$-\frac{ig}{\cos\theta_W}\,\gamma^\mu(t_3\mathsf{L} - Q\sin^2\theta_W)\,. \tag{15.41}$$

□ **Exercise 15.6** Show that the gauge interactions of the fermions violate parity and C-invariance, but not CP.

15.3.2 Electron mass

So far, we have discussed the gauge covariant derivative terms involving the fermions, which include the kinetic term for free fermions and the gauge interactions. We now discuss the mass of the fermions.

For any fermion field ψ, the mass term can be written as

$$\mathscr{L}_{\text{mass}} = -m\overline{\psi}\psi = -m\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right)\,. \tag{15.42}$$

Take the electron field, for example. The e_L transforms like part of a doublet under the SU(2) part of the gauge group, and the e_R transforms like a singlet. The combination of the two cannot be gauge invariant. Thus, in the fundamental Lagrangian of the standard model, there is no mass term for the electron.

However, the Lagrangian has spontaneous symmetry breaking, and to drive this process, we had to introduce a doublet of Higgs bosons. Leptons can have gauge invariant interactions involving the Higgs bosons. These are Yukawa-type interactions, given by

$$\mathscr{L}_{\text{Yuk}} = -h_e\left(\overline{\Psi}_{eL}\phi e_R + \bar{e}_R\phi^\dagger\Psi_{eL}\right)\,, \tag{15.43}$$

where h_e is a coupling constant.

□ **Exercise 15.7** Check that the interaction terms of Eq. (15.43) are invariant under the gauge group $SU(2) \times U(1)$.

After spontaneous symmetry breaking, if we rewrite the Higgs field ϕ in terms of the quantum fields defined in Eq. (15.16), we obtain

$$\mathscr{L}_{\text{Yuk}} = -h_e \Big[\frac{v}{\sqrt{2}} (\bar{e}_L e_H + \bar{e}_R e_L) + \bar{\nu}_{eL} e_R \phi^+ + \bar{e}_R \nu_{eL} \phi^- + \frac{1}{\sqrt{2}} (\bar{e} e H + i \bar{e} \gamma_5 e \zeta) \Big] . \tag{15.44}$$

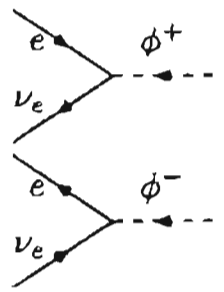
The terms involving the vacuum expectation value v of the Higgs field are the mass terms for the electron field, which have been generated by the spontaneous symmetry breaking process. Comparing with Eq. (15.42), we find that the electron mass is given in this model in terms of the Yukawa coupling constant h_e and the vacuum expectation value of the Higgs boson field:

$$m_e = \frac{h_e v}{\sqrt{2}} . \tag{15.45}$$

The neutrino, on the other hand, remains massless even after spontaneous symmetry breaking because of the absence of the ν_{eR}.

15.3.3 Yukawa couplings

The Yukawa couplings, i.e., the couplings of fermions with scalar fields of the theory, appear in Eq. (15.44). We can rewrite them using Eq. (15.45). For the unphysical charged scalar modes, this gives the following Feynman rules:

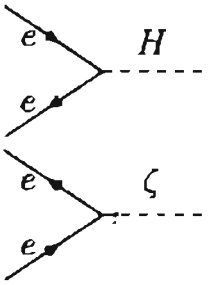


$$: \quad -i \frac{\sqrt{2} m_e}{v} \mathsf{R} = -i \frac{g m_e}{\sqrt{2} M_W} \mathsf{R} ,$$

$$: \quad -i \frac{\sqrt{2} m_e}{v} \mathsf{L} = -i \frac{g m_e}{\sqrt{2} M_W} \mathsf{L} , \tag{15.46}$$

where in both cases we have used Eq. (15.19) in writing the latter form. Similarly, the neutral scalar modes have the following cou-

plings:



$$
\begin{aligned}
&: \quad -i\,\frac{m_e}{v} = -i\,\frac{g}{2\cos\theta_W}\,\frac{m_e}{M_Z}\,, \\
&: \quad \frac{m_e}{v}\,\gamma_5 = \frac{g}{2\cos\theta_W}\,\frac{m_e}{M_Z}\,.
\end{aligned}
\tag{15.47}
$$

The neutrino has no coupling with the neutral scalar modes since the model has no right-chiral neutrino.

15.3.4 Other fermions in the model

It is now known that there are three generations of leptons. In the standard model, the second and the third generations of leptons look exactly the same as the first one. The charged leptons in the second and the third generations are the muon and the tau. We can write all the couplings involving these particles by looking at what we have done for the electron and merely changing the electron mass to the relevant lepton mass. There is also a neutrino corresponding to each charged lepton.

If we try to incorporate quarks in the model, we find similarities and differences with leptons. Like leptons, quarks also come in three generations. In any one generation, the left-chiral components of quarks come in SU(2) doublets and the right-chiral ones in singlets. The representations are

$$
\begin{aligned}
q_{AL} \equiv \begin{pmatrix} u_{AL} \\ d_{AL} \end{pmatrix} &: (2, \frac{1}{6})\,, \\
u_{AR} &: (1, \frac{2}{3})\,, \\
d_{AR} &: (1, -\frac{1}{3})\,.
\end{aligned}
\tag{15.48}
$$

Here A is a generation index that runs from 1 to 3. Collectively, we will call all the u_A's as up-type quarks and d_A's as down-type quarks. Since all generations have the same group transformation property, the Yukawa couplings can connect any two generations. So for the quarks,

$$
\mathscr{L}_{\text{Yuk}} = -\sum_{A,B} \left(h_{AB}\bar{q}_{AL}\phi d_{BR} + h'_{AB}\bar{q}_{AL}\tilde{\phi}u_{BR} + \text{h.c.} \right), \tag{15.49}
$$

where $\tilde{\phi} = i\tau_2\phi^*$. We now see the differences with the leptonic sector, which we discuss in some detail.

Here we have two kinds of terms. In the leptonic sector, we had only the first kind since there are no right handed neutrinos in the model. The right chiralities of the up-type quarks now give rise to the second term. In constructing this, we have used $\tilde{\phi}$ defined above, which contains the complex conjugates of the fields appearing in the multiplet ϕ:

$$\tilde{\phi} \equiv \begin{pmatrix} \phi_0^* \\ -\phi^- \end{pmatrix} \; : \; (2, -\frac{1}{2}) . \tag{15.50}$$

□ **Exercise 15.8** If ϕ is any SU(2) doublet, show that ϕ^* does not transform like a doublet, but $\tilde{\phi} = i\tau_2\phi^*$ does.

So far, there does not seem to be any problem. We can of course have Yukawa couplings involving two different fermions. The problem arises when we replace ϕ by the quantum fields using Eq. (15.16). This will now give mass terms as in Eq. (15.44), but they will involve two different fermions in general.

We can, of course, partly avoid the problem by choosing the basis states of Eq. (15.48) in such a way that the 3×3 matrix h' is diagonal. This implies choosing a basis in which u_1, u_2, u_3 are really the physical particles, the up-quark u, the charm quark c and the top quark t. However, the matrix h need not be diagonal in this basis. This is a new feature, not encountered for leptons, since there was only one kind of coupling in the leptonic case which could always be taken as diagonal. In the present case, we obtain a *mass matrix* for the down-type quarks, given by

$$m_{AB} = \frac{h_{AB}v}{\sqrt{2}} . \tag{15.51}$$

To obtain the eigenstates of the down-type quarks, we then need to diagonalize this mass matrix. This can be done in general by using a bi-unitary transformation. It means that for any matrix m, it is possible to find two unitary matrices K and K' such that $\overline{m} \equiv K'mK^\dagger$ is diagonal, with non-negative entries. These entries can then be interpreted as the masses of the physical down-type quarks, which we identify with the down quark d, the strange quark s and the bottom quark b. We will denote these by $\mathscr{D}_A$ with $A = 1, 2, 3$.

The mass terms of these quarks coming from Eq. (15.49) can then be written as

$$\sum_{A,B} (K \overline{m} K'^{\dagger})_{AB} \, \bar{d}_{AL} d_{BR} \,. \tag{15.52}$$

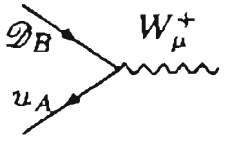
This shows that the eigenstates can be defined as

$$\mathscr{D}_{AL} = K^{\dagger}_{AB} d_{BL} \,, \qquad \mathscr{D}_{AR} = K'^{\dagger}_{AB} d_{BR} \,. \tag{15.53}$$

This has an interesting consequence on the gauge interactions of quarks. The interaction with the W-bosons can be written as in Eq. (15.33), where one must use the components of the same gauge multiplets. But since the physical down-type quarks are different from those in the multiplets, we get

$$\begin{aligned} \mathscr{L}^{(W)}_{\rm int} &= -\frac{g}{\sqrt{2}} \sum_{A} \left(\bar{u}_{AL} \gamma^{\mu} d_{AL} W^{+}_{\mu} + \text{h.c.} \right) \\ &= -\frac{g}{\sqrt{2}} \sum_{A,B} \left(\bar{u}_{AL} \gamma^{\mu} K_{AB} \mathscr{D}_{BL} W^{+}_{\mu} + \text{h.c.} \right) \,, \end{aligned} \tag{15.54}$$

using Eq. (15.53) to express everything in terms of the eigenstates. The matrix K is called the quark mixing matrix, or the Cabibbo-Kobayashi-Maskawa matrix. The Feynman rule for the W coupling for the quarks is therefore



$$: \quad -\frac{ig}{\sqrt{2}} K_{AB} \gamma^{\mu} \mathsf{L} \,. \tag{15.55}$$

□ **Exercise 15.9** Find the couplings of the Z bosons with the quark eigenstates. Show that these do not involve the mixing matrix K, and is given by Eq. (15.41).

15.4 Gauge boson decay

We have discussed processes involving photons in Ch. 9. Photons are massless, and hence cannot decay. In the electroweak theory, we have massive gauge bosons W and Z, which can decay. This is a new feature of the electroweak theory, which we discuss now.

To be specific, let us consider the decay of the Z-boson to a fermion-antifermion pair. The coupling depends on which fermion

we are talking about. In order to be general, let us write the coupling as

$$\mathscr{L}_{\rm int} = -\frac{g}{2\cos\theta_W}\,\overline{f}\gamma^\mu(a_f - b_f\gamma_5)f\,Z_\mu\,, \tag{15.56}$$

where

$$a_f = t_3 - 2Q\sin^2\theta_W\,, \qquad b_f = t_3\,, \tag{15.57}$$

which follow from Eq. (15.41). The Feynman amplitude obtained from this would be

$$\mathscr{M} = -\frac{ig}{2\cos\theta_W}\,\overline{u}(p)\gamma^\mu(a_f - b_f\gamma_5)v(p')\,\epsilon_\mu(k)\,, \tag{15.58}$$

where p and p' denote the momenta of the fermion and the antifermion in the final state, and $\epsilon_\mu(k)$ is the polarization vector of the decaying Z-boson.

We want the decay of unpolarized Z bosons. For this we need to average over the initial polarizations and sum over the final spins. This gives

$$\begin{aligned}\overline{|\mathscr{M}|^2} &= \frac{g^2}{4\cos^2\theta_W}\,\langle\epsilon_\mu(k)\epsilon_\nu^*(k)\rangle \\ &\quad\times \mathrm{Tr}\left[\not{p}\gamma^\mu(a_f - b_f\gamma_5)\not{p}'\gamma^\nu(a_f - b_f\gamma_5)\right].\end{aligned} \tag{15.59}$$

In writing this expression, we have neglected the fermion masses since the Z-boson is much heavier than all known fermions which it can decay to. Thus we obtain

$$\begin{aligned}\overline{|\mathscr{M}|^2} &= \frac{g^2}{\cos^2\theta_W}\,\langle\epsilon_\mu(k)\epsilon_\nu^*(k)\rangle \\ &\quad\times (a_f^2 + b_f^2)(p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}p\cdot p')\,.\end{aligned} \tag{15.60}$$

There is also an antisymmetric contribution to the trace, but it does not contribute in the final count since the polarization average is symmetric in the Lorentz indices, as we will now see.

Polarization sum

In Ch. 8, we showed that the photon has only two physical degrees of freedom and calculated the polarization sum that can be used for

external photon legs in Eq. (8.88). For a massive gauge boson like the Z, this expression cannot be used for two reasons. First, k^2 is not zero for a massive on-shell particle with momentum k. Second, an arbitrary massive gauge boson need not couple to conserved currents.

Since a spin-1 boson cannot have more than three internal degrees of freedom, we can always choose the physical polarization vectors so that each of them satisfies the relation

$$k^\mu \epsilon_\mu(k) = 0 \tag{15.61}$$

and are normalized by

$$\epsilon_r^\mu(k)\epsilon_{s\mu}(k)^* = -\delta_{rs}\,. \tag{15.62}$$

In the case of the photon, only two such vectors are allowed for reasons described in Ch. 8. But in the case of a massive gauge boson, there can be three of them. For the 3-momentum $\boldsymbol{k}$ and energy $E = \sqrt{\mathbf{k}^2 + M^2}$, these are the two unit spatial vectors orthogonal to $\boldsymbol{k}$, and a third one given by

$$\epsilon_l^\mu(k) = \frac{1}{M}\left(\mathbf{k}, E\,\widehat{k}\right)\,, \tag{15.63}$$

where $\widehat{k}$ is the unit vector along $\boldsymbol{k}$. This is usually called the *longitudinal polarization vector* since its spatial components are in the same direction as $\boldsymbol{k}$, and so we have denoted it with the subscript 'l'. The transverse polarization vectors, on the other hand, can be chosen just as for the photons. Combining, we find that the three physical polarization vectors satisfy the relation

$$\sum_r \epsilon_{r\mu}(k)\epsilon_{r\nu}^*(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}\,. \tag{15.64}$$

To obtain the polarization average, we need to divide this by the number of polarization states, i.e., by 3. We see that the polarization average is symmetric in the Lorentz indices.

Decay rate

Going back to Eq. (15.60) now, we can write the matrix element square as

$$\begin{aligned}\overline{|\mathscr{M}|^2} &= \frac{g^2}{3\cos^2\theta_W}\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{M_Z^2}\right)\\ &\quad\times (a_f^2+b_f^2)(p^\mu p'^\nu + p^\nu p'^\mu - g'^{\mu\nu}p\cdot p')\\ &= \frac{g^2}{3\cos^2\theta_W}(a_f^2+b_f^2)\left(p\cdot p'+\frac{2p\cdot k\, p'\cdot k}{M_Z^2}\right). \quad (15.65)\end{aligned}$$

The rest frame of the Z-boson is defined by

$$k^\mu = (M_Z, \mathbf{0}). \quad (15.66)$$

In this frame, we can write

$$p^\mu = (\tfrac{1}{2}M_Z, \boldsymbol{p}), \qquad p'^\mu = (\tfrac{1}{2}M_Z, -\boldsymbol{p}), \quad (15.67)$$

where $\mathbf{p} = \frac{1}{2}M_Z$ since the masses of the fermions are being neglected. Using these, we obtain

$$\overline{|\mathscr{M}|^2} = \frac{g^2M_Z^2}{3\cos^2\theta_W}(a_f^2+b_f^2). \quad (15.68)$$

Plugging this into Eq. (7.119) and performing the angular integrations, we find the decay rate to be

$$\begin{aligned}\Gamma(Z\to f\bar{f}) &= \frac{g^2M_Z}{48\pi\cos^2\theta_W}(a_f^2+b_f^2)\\ &= \frac{\alpha M_Z}{12\sin^2\theta_W\cos^2\theta_W}(a_f^2+b_f^2), \quad (15.69)\end{aligned}$$

using Eq. (15.40) in the last step. Note that this is the decay rate into a specific pair of decay products. To find the total decay rate of the Z-boson, one needs to sum the rates over all possible final states.

□ **Exercise 15.10** Using $M_Z = 91\,\text{GeV}$, show that the decay rate into a neutrino-antineutrino pair is about 165 MeV. Use $\sin^2\theta_W = 0.23$.

□ **Exercise 15.11** Using $M_W = 80\,\text{GeV}$, show that the rate for the decay $W^+ \to e^+\nu_e$ is about 220 MeV. You can neglect the mass of the electron.

□ **Exercise 15.12** Consider the decay of the Higgs boson H into a lepton-antilepton pair $\ell^+\ell^-$ (where $\ell = e^-, \mu^-$ etc). Show that the decay rate is given by

$$\Gamma(H \to \ell^+\ell^-) = \frac{\alpha m_\ell^2 M_H}{8 \sin^2\theta_W \cos^2\theta_W M_Z^2} \left(1 - \frac{4m_\ell^2}{M_H^2}\right)^{3/2}. \quad (15.70)$$

[Hint: Have you seen this done earlier in the book in some form?]

15.5 Scattering processes

15.5.1 Forward-backward asymmetry

In §9.6, we discussed the process $e^-e^+ \to \mu^-\mu^+$ in QED. We can now calculate how the cross section is modified by weak corrections.

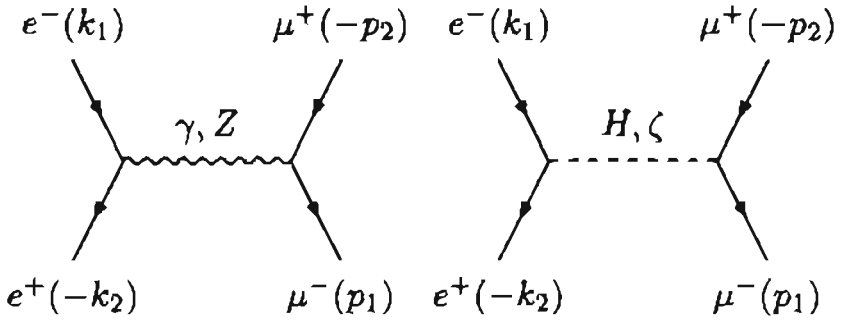


Figure 15.2: Lowest order diagrams for the process $e^-e^+ \to \mu^-\mu^+$ in the electroweak theory.

There are four diagrams in the lowest order, as shown in Fig. 15.2. Of these, we have already calculated the amplitude for the photon mediated diagram in §9.6 and found

$$\mathscr{M}_\gamma = \frac{e^2}{s}\left[\overline{v}(k_2)\gamma^\lambda u(k_1)\right]\left[\overline{u}(p_1)\gamma_\lambda v(p_2)\right], \quad (15.71)$$

where s is the Mandelstam variable. Notice also that the diagrams mediated by the H and ζ contain couplings which are proportional to the fermion masses. It was already argued in §9.6 that it is perfectly justifiable to neglect the electron mass in studying this process. Hence these two diagrams do not contribute to the amplitude in this approximation. The coupling of the electron and the muon with the Z-boson are the same, given in Eq. (15.39). Using Eq. (15.40), we

can write the Feynman rule in the form

$$-\frac{ie}{\sin 2\theta_W}\gamma_\mu(g_V - g_A\gamma_5)\,, \tag{15.72}$$

where

$$g_V = -\frac{1}{2} + 2\sin^2\theta_W\,, \qquad g_A = -\frac{1}{2}\,. \tag{15.73}$$

Then we can write the Z-mediated amplitude as

$$\begin{aligned}\mathscr{M}_Z = \frac{e^2 f_Z}{s}&\left[\overline{v}(k_2)\gamma^\lambda(g_V - g_A\gamma_5)u(k_1)\right]\\ &\times\left[\overline{u}(p_1)\gamma_\lambda(g_V - g_A\gamma_5)v(p_2)\right],\end{aligned} \tag{15.74}$$

by choosing the $\xi = 1$ gauge, and defining

$$f_Z = \frac{s}{\sin^2 2\theta_W(s - M_Z^2)}\,. \tag{15.75}$$

So the square of the total amplitude, averaged over initial spins and summed over final ones, is given by

$$\overline{|\mathscr{M}|^2} = \frac{1}{4}\sum_{\text{spin}}\left|\mathscr{M}_\gamma + \mathscr{M}_Z\right|^2. \tag{15.76}$$

There are four terms, one of which has been calculated in §9.6 and found to be

$$\overline{|\mathscr{M}_\gamma|^2} = e^4\left(1 + \cos^2\theta\right)\,, \tag{15.77}$$

where now we neglect the muon mass as well, anticipating that the corrections from the Z-boson can be important only if s is comparable to M_Z^2, and therefore much higher than m_μ^2. The square of the Z-mediated amplitude is given by

$$\overline{|\mathscr{M}_Z|^2} = e^4 f_Z^2\left[(g_V^2 + g_A^2)^2(1 + \cos^2\theta) + 8g_A^2 g_V^2\cos\theta\right]. \tag{15.78}$$

As for the cross terms, we find

$$\begin{aligned}\overline{\mathscr{M}_\gamma^*\mathscr{M}_Z} = \frac{e^4 f_Z}{s^2}&\mathrm{Tr}\left[\not{k}_2\gamma^\lambda(g_V - g_A\gamma_5)\not{k}_1\gamma^\rho\right]\\ &\times\mathrm{Tr}\left[\not{p}_1\gamma_\lambda(g_V - g_A\gamma_5)\not{p}_2\gamma_\rho\right].\end{aligned} \tag{15.79}$$

The $g_A g_V$ terms vanish because they multiply one trace involving γ_5 which is antisymmetric in the Lorentz indices λ, ρ, and another trace which is symmetric in these indices. We are then left with

$$\overline{\mathscr{M}_\gamma^* \mathscr{M}_Z + \text{h.c.}} = 2e^4 f_Z \left[g_V^2 (1 + \cos^2\theta) + 2 g_A^2 \cos\theta \right]. \quad (15.80)$$

Adding up all the contributions, we can write

$$\overline{|\mathscr{M}|^2} = e^4 \left[(1 + a_1)(1 + \cos^2\theta) + a_2 \cos\theta \right], \quad (15.81)$$

where

$$\begin{aligned} a_1 &= 2 f_Z g_V^2 + f_Z^2 (g_V^2 + g_A^2)^2 , \\ a_2 &= 4 f_Z g_A^2 + 8 f_Z^2 g_V^2 g_A^2 . \end{aligned} \quad (15.82)$$

The a_1 contribution has the same angular dependence as the pure QED result. The important qualitative difference from the pure QED result is the appearance of the a_2-term, which is proportional to $\cos\theta$. Without this term, the differential cross section would be invariant under the transformation $\theta \to \pi - \theta$. In other words, it would have been forward-backward symmetric. The a_2-term introduces a forward-backward asymmetry.

The forward-backward asymmetry is in general defined by the ratio

$$\mathcal{A} = \frac{\displaystyle\int_0^{\pi/2} d\theta \sin\theta \frac{d\sigma}{d\Omega} - \int_{\pi/2}^{\pi} d\theta \sin\theta \frac{d\sigma}{d\Omega}}{\displaystyle\int_0^{\pi} d\theta \sin\theta \frac{d\sigma}{d\Omega}} , \quad (15.83)$$

with the integration over the azimuthal angle implied for each integral. For this process, Eq. (15.81) gives

$$\mathcal{A} = \frac{3}{8} \frac{a_2}{1 + a_1} . \quad (15.84)$$

□ **Exercise 15.13** Evaluate the amount of forward-backward asymmetry in this process at $s = \frac{1}{4} M_Z^2$, assuming $\sin^2\theta_W = 0.23$.

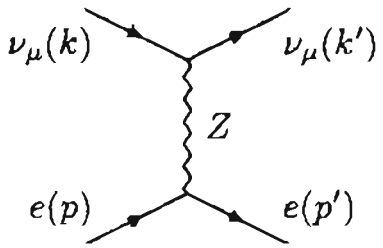


Figure 15.3: Lowest order diagram for the process $\nu_\mu e \to \nu_\mu e$ in the electroweak theory.

15.5.2 Low energy weak interactions

The process we discussed above can be mediated by electromagnetic as well as by weak interaction. Now we consider processes which can go only via the mediation of the W or the Z. These will be purely weak processes.

As an example, we take the elastic scattering of muon-neutrinos (ν_μ) by electrons. This can be mediated by the Z-boson exchange, as shown in Fig. 15.3. There is no diagram with neutral scalar exchange because, as we saw in §15.3, the neutrinos do not couple to neutral scalars.

The Feynman amplitude of this process can be directly written using the Feynman rules given in Eqs. (15.36) and (15.39):

$$i\mathscr{M} = -\left(\frac{ig}{2\cos\theta_W}\right)^2 \frac{-ig^{\lambda\rho}}{q^2 - M_Z^2} \times \left[\overline{u}(k')\gamma_\lambda \mathsf{L} u(k)\right]\left[\overline{u}(p')\gamma_\rho(g_V - g_A\gamma_5)u(p)\right], \quad (15.85)$$

where we have used the Feynman-'t Hooft gauge and introduced the notation $q = k - k' = p' - p$.

Let us suppose now that the energies involved in the process are much smaller than M_Z. Then $q^2 \ll M_Z^2$, and we can neglect the momentum dependence of the Z propagator to write

$$\mathscr{M} = \left(\frac{g}{2M_Z\cos\theta_W}\right)^2 \times \left[\overline{u}(k')\gamma_\lambda \mathsf{L} u(k)\right]\left[\overline{u}(p')\gamma^\lambda(g_V - g_A\gamma_5)u(p)\right]. \quad (15.86)$$

But this is exactly the amplitude that we would have obtained if we

had started from a 4-fermion interaction Lagrangian

$$\mathscr{L}_{\rm int} = \frac{G_F}{\sqrt{2}}\left[\overline{\psi}_{(\nu_\mu)}\gamma_\lambda(1-\gamma_5)\psi_{(\nu_\mu)}\right]\left[\overline{\psi}_{(e)}\gamma^\lambda(g_V - g_A\gamma_5)\psi_{(e)}\right]. \quad (15.87)$$

provided we had identified the Fermi constant as

$$G_F = \frac{g^2}{4\sqrt{2}M_Z^2\cos^2\theta_W} = \frac{g^2}{4\sqrt{2}M_W^2}, \quad (15.88)$$

using Eq. (15.22) in the last step. Thus at low energies, weak interaction is given by 4-fermion interactions.

In this limit therefore, the calculation of the cross section is very similar to what was shown in §7.5. We omit the details and write directly

$$\overline{|\mathscr{M}|^2} = 16G_F^2\Big[(g_V+g_A)^2(k\cdot p)^2 + (g_V-g_A)^2(k'\cdot p)^2 - (g_V^2 - g_A^2)m_e^2 k\cdot k'\Big] \quad (15.89)$$

for this case. Then, in the Lab frame in which the initial electron is at rest, the differential cross section can be written down using Eq. (7.118):

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 m_e^2}{4\pi^2}\frac{1}{(m_e+\omega-\omega\cos\theta)^2} \times\Big[(g_V+g_A)^2\omega^2 + (g_V-g_A)^2\omega'^2 - (g_V^2-g_A^2)\omega\omega'(1-\cos\theta)\Big]. \quad (15.90)$$

Here θ is the angle between the initial and the final neutrino momenta, and

$$\omega' = \frac{m_e\omega}{m_e+\omega-\omega\cos\theta} \quad (15.91)$$

is the energy of the final neutrino. The total cross section can be found by integrating over the angular variables. For $\omega \gg m_e$ one obtains

$$\sigma = \frac{G_F^2 m_e\omega}{2\pi}\left[(g_V+g_A)^2 + \frac{1}{3}(g_V-g_A)^2\right]. \quad (15.92)$$

□ **Exercise 15.14** The muon decay, $\mu^- \to e^- + \widehat{\nu}_e + \nu_\mu$, proceeds via a W-exchange graph. Draw the diagram and write down the amplitude. Now use Eq. (15.88) to show that if the energies are small compared to the W-mass, the lowest order amplitude is the same as that obtained from the 4-fermion Lagrangian of Eq. (7.36).

□ **Exercise 15.15** For the elastic scattering between the ν_e and the electron, there is an additional diagram with a W-boson exchange which contributes at this order. Add the amplitude of this diagram with that of the Z-mediated diagram, and show that $|\overline{\mathcal{M}}|^2$ is given by the same form as in Eq. (15.89) provided we define

$$g_V = \frac{1}{2} + 2\sin^2\theta_W\,, \qquad g_A = \frac{1}{2}\,. \tag{15.93}$$

15.5.3 High energy scattering

As a last example of scattering processes in the standard electroweak theory, we consider the pair creation of W bosons:

$$e^-(p_-) + e^+(p_+) \to W^-(k_-) + W^+(k_+)\,. \tag{15.94}$$

The initial center of mass energy has to be high enough to produce the W's in the final state. Therefore we can neglect the electron mass.

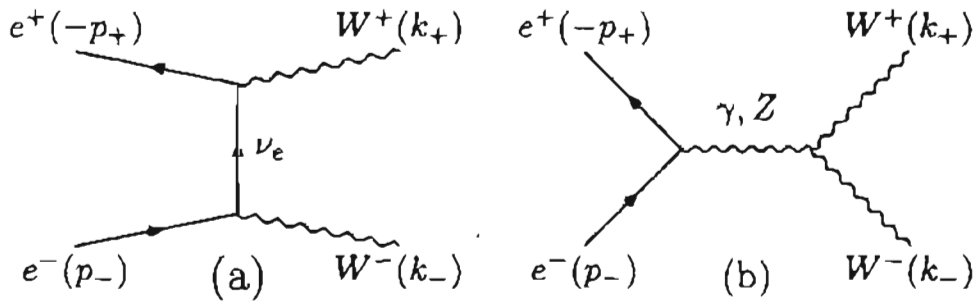


Figure 15.4: Lowest order diagrams for $e^-e^+ \to W^-W^+$.

There are five diagrams for this process at the tree level. One of them, shown in Fig. 15.4a, has an intermediate neutrino line. Two others are collectively shown in Fig. 15.4b, where the internal line can be either a photon or a Z-boson. In addition, there are diagrams involving exchanges of neutral scalars H and ζ.

The calculation is very lengthy for unpolarized particles in the initial as well as in the final states. So we will demonstrate this calculation in a very specific case, characterized by the following conditions:

1. The electrons have positive helicity.

2. Only the production of longitudinal polarization states of the W's is considered.

3. The center of mass energy is much higher than M_W, i.e., $s \gg M_W^2$.

Since we are neglecting the electron mass, the diagrams involving neutral scalar exchange vanish. In addition, the first assumption means that the electrons are right chiral. In this case, Fig. 15.4a does not contribute either. This can be understood from the coupling given in Eq. (15.34), which shows that it vanishes for a right-chiral electron. Since the couplings involve a γ-matrix, it also implies that the positron has to be right-chiral as well.

The second assumption means that the polarization vector for the final state particles can be chosen as in Eq. (15.63). Moreover, since $E = \mathbf{k}$ due to the third assumption, we can write the longitudinal polarization vector simply as

$$\epsilon^\mu(k) \approx \frac{k^\mu}{M_W} . \tag{15.95}$$

These considerations allow us to write down the amplitudes in the Feynman-'t Hooft gauge as

$$\begin{aligned} i\mathscr{M}_\gamma &= \left[\overline{v}_R(p_+) ie\gamma_\rho u_R(p_-)\right] \frac{-ig^{\lambda\rho}}{s} \\ &\quad \times ieV_{\mu\nu\lambda}(k_-, k_+)\, \epsilon_-^\mu(k_-)\epsilon_+^\nu(k_+) , \\ i\mathscr{M}_Z &= \left[\overline{v}_R(p_+) \frac{ig}{2\cos\theta_W} \gamma_\rho(\mathsf{L} - 2\sin^2\theta_W) u_R(p_-)\right] \frac{-ig^{\lambda\rho}}{s - M_Z^2} \\ &\quad \times ig\cos\theta_W V_{\mu\nu\lambda}(k_-, k_+)\, \epsilon_-^\mu(k_-)\epsilon_+^\nu(k_+) , \end{aligned} \tag{15.96}$$

where $\epsilon_\pm$ are the longitudinal polarization vectors for the $W^\pm$, and

$$V_{\mu\nu\lambda}(k_-, k_+) = (k_- - k_+)_\lambda g_{\mu\nu} + (2k_+ + k_-)_\mu g_{\nu\lambda} - (2k_- + k_+)_\nu g_{\lambda\mu} . \tag{15.97}$$

□ **Exercise 15.16** Use the Feynman rule for cubic coupling from Fig. 14.1 and the definitions of the Z boson and the photon in Eq. (15.20) to verify the WWZ and WWA couplings used above.

Since $u_R = Ru$ where $R = \frac{1}{2}(1 + \gamma_5)$, the left-chiral projector in the Z-coupling does not contribute. Then, using Eq. (15.40), we can

combine the two amplitudes to write

$$\mathscr{M} = \mathscr{M}_\gamma + \mathscr{M}_Z = e^2\Big[\bar{v}_R(p_+)\gamma^\lambda u_R(p_-)\Big]\left(\frac{1}{s} - \frac{1}{s - M_Z^2}\right) \times V_{\mu\nu\lambda}(k_-, k_+)\,\epsilon_-^\mu(k_-)\epsilon_+^\nu(k_+)\,. \tag{15.98}$$

If we now use Eq. (15.95), we obtain

$$V_{\mu\nu\lambda}(k_-, k_+)\,\epsilon_-^\mu(k_-)\epsilon_+^\nu(k_+) = \frac{1}{M_W^2}(k_+ \cdot k_- + M_W^2)(k_+ - k_-)_\lambda \approx \frac{s}{2M_W^2}(k_+ - k_-)_\lambda\,, \tag{15.99}$$

using $s \gg M_W^2$ in the last step to write $k_+ \cdot k_- = \frac{1}{2}s$ in the center of mass frame. The factor in Eq. (15.98) coming from the propagators can also be simplified as

$$\frac{1}{s} - \frac{1}{s - M_Z^2} = -\frac{M_Z^2}{s(s - M_Z^2)} \approx -\frac{M_Z^2}{s^2}\,. \tag{15.100}$$

Putting all of these together, we finally obtain the amplitude to be

$$\mathscr{M} = -\frac{e^2}{2s\cos^2\theta_W}\Big[\bar{v}_R(p_+)\gamma^\lambda u_R(p_-)\Big](k_+ - k_-)_\lambda\,, \tag{15.101}$$

using the relation between the masses of the W and the Z bosons from Eq. (15.22).

□ **Exercise 15.17** ⋆ Show that one obtains exactly the same amplitude as in Eq. (15.101) if the external $W^\pm$-lines are imagined to be replaced by the $\phi^\pm$ lines. You need to derive the Feynman rules for the ϕ couplings to calculate this diagram. [Note: This equivalence of the amplitude holds only for the longitudinal polarization state at energies much higher than the gauge boson mass, and is known as the *equivalence theorem*.]

There is now no average over initial spin since we have taken the initial fermions in specific spin states. So we cannot directly use the spin sum formulas while calculating the square of this matrix element. Instead, we use the chirality projection operators to write

$$\Big[\bar{v}_R(p_+)\gamma^\lambda u_R(p_-)\Big] = \Big[\bar{v}(p_+)\gamma^\lambda R u(p_-)\Big]\,. \tag{15.102}$$

It then follows that

$$\begin{aligned}
&\Big[\overline{v}_R(p_+)\gamma^\rho u_R(p_-)\Big]^\dagger \Big[\overline{v}_R(p_+)\gamma^\lambda u_R(p_-)\Big] \\
&\quad = \Big[\overline{u}(p_-)\gamma^\rho R v(p_+)\Big]\Big[\overline{v}(p_+)\gamma^\lambda R u(p_-)\Big] \\
&\quad = \sum_{\text{spin}} \Big[\overline{u}(p_-)\gamma^\rho R v(p_+)\Big]\Big[\overline{v}(p_+)\gamma^\lambda R u(p_-)\Big] \,. \qquad (15.103)
\end{aligned}$$

The last equality works because of the projector appearing in the bilinears. Even if we sum over all spins, the projector will pick out only the corresponding projection. This allows us to use the spin sum technique to write the above expression as

$$\text{Tr}\left[\not{p}_+\gamma^\lambda R \not{p}_-\gamma^\rho R\right] = 2\Big(p_+^\lambda p_-^\rho + p_-^\lambda p_+^\rho - g^{\lambda\rho} p_+\cdot p_-\Big) + \varepsilon\text{-term}\,. \qquad (15.104)$$

The quantity denoted by 'ε-term' contains the antisymmetric tensor, and is irrelevant since the expression obtained here is contracted with something which is symmetric in λ, ρ. So we get

$$\begin{aligned}
|\mathscr{M}|^2 &= \frac{e^4}{2s^2\cos^4\theta_W}\Big(p_+^\lambda p_-^\rho + p_-^\lambda p_+^\rho - \frac{1}{2}s g^{\lambda\rho}\Big)(k_+ - k_-)_\lambda (k_+ - k_-)_\rho \\
&= \frac{e^4}{4\cos^4\theta_W}\,\sin^2\theta\,. \qquad (15.105)
\end{aligned}$$

Here θ is the scattering angle, which we have taken to be the angle between $\boldsymbol{p}_-$ and $\boldsymbol{k}_-$. We have also used kinematical relations like $p_+\cdot p_- = k_+\cdot k_- = \frac{1}{2}s$ which are valid with our assumptions.

The scattering cross section can now easily be obtained using Eq. (7.106):

$$\sigma = \frac{\pi\alpha^2}{6s\cos^4\theta_W}\,. \qquad (15.106)$$

□ **Exercise 15.18** Consider the decay of muons at rest with spins aligned along the direction $\hat{s}$. With the interaction of Eq. (7.36) and using spin projection operators, show that $\overline{|\mathscr{M}|^2}$ is obtained by replacing p^λ by $p^\lambda - m_\mu n^\lambda$ in Eq. (7.49), where n^λ is defined by Eq. (4.81). Assuming $m_e = 0$, show that the differential decay is given by

$$\frac{d\Gamma}{d\Omega} = \frac{G_F^2 m_\mu^5}{768\pi^4}\left(1 - \frac{1}{3}\cos\theta\right). \qquad (15.107)$$

where θ is the angle between $\hat{s}$ and the electron momentum. [Note: Under parity $\theta \to \pi - \theta$ since the spin remains invariant while the momenta are reversed. The $\cos\theta$ term then signals parity violation.]

15.6 Propagator for unstable particles

Clearly, the scattering formulas derived in §15.5 do not hold if the incident particle energies are such that $s = M_Z^2$. For example, the quantity f_Z, defined in Eq. (15.75), blows up. This is because the Z-boson is an unstable particle. In writing the propagator of any unstable particle, we must take its non-zero decay width into account.

To see how this can be done, let us consider a scalar field for the sake of simplicity. For a stable particle, the solutions of the free equation of motion are plane waves. If however the particle has a decay width Γ in its rest frame, the free particle solutions $\phi(x)$ should decay with time as

$$|\phi(x)|^2 \sim \exp\left(-\frac{m}{E}\Gamma t\right) \tag{15.108}$$

up to normalization factors. The factor m/E occurs in a general frame to account for the time dilation factor for its lifetime $1/\Gamma$. Thus we can write

$$\phi(x) \sim \exp\left(-iEt + i\boldsymbol{p}\cdot\boldsymbol{x} - \frac{m}{2E}\Gamma t\right) . \tag{15.109}$$

In order to identify a particle by its energy, the uncertainty in energy measurements must be much less than its energy, i.e., $\Delta E \ll E$. In the rest frame of the particle, this implies $\Delta E \ll m$. Due to the time-energy uncertainty relation, the time taken to make the measurement will satisfy $\Delta t \gg 1/m$. Unless this time is much less than the lifetime $1/\Gamma$ of the particle, no such measurement would be possible. Thus for a particle interpretation of the quanta of the field, it is necessary that the condition $\Gamma \ll m$ and therefore $\Gamma \ll E$ be satisfied. In that case, ignoring $O(\Gamma^2/m^2)$ corrections, the equation of motion for the field $\phi(x)$ can be written from Eq. (15.109) as

$$\left(\Box + m^2 - im\Gamma\right)\phi(x) = 0 . \tag{15.110}$$

We can now use the methods of §3.7 to find the propagator of this field. We will get

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + im\Gamma} . \tag{15.111}$$

A similar analysis for vector bosons would give the following propagator for an unstable vector boson in the Feynman-'t Hooft gauge:

$$D_{\mu\nu}(p) = -\frac{g_{\mu\nu}}{p^2 - m^2 + im\Gamma}\,. \tag{15.112}$$

Since $\Gamma \ll m$ by our assumption, the extra term added is immaterial unless p^2 is very close to m^2, i.e., unless the particle is on-shell, or very nearly so. But for a nearly on-shell particle in the intermediate state, one should use this propagator.

□ **Exercise 15.19** Find the forward-backward asymmetry in $e^-e^+ \to \mu^-\mu^+$ at $s = M_Z^2$. Take $\sin^2\theta_W = 0.23$ as before, and $\Gamma_Z = 2.49\,\text{GeV}$.

15.7 Global symmetries of the model

The standard electroweak model is based on a local symmetry with the gauge group SU(2) × U(1). This of course entails the same global symmetry. But in addition, the standard model has some other global symmetries as well.

One of these is the baryon number symmetry, which is defined as

$$q \to e^{-i\mathcal{B}\theta} q\,, \tag{15.113}$$

where q here stands for all quark fields, and $\mathcal{B}$ is the *baryon number* of quarks, usually taken to be $\frac{1}{3}$. Antiquarks have $\mathcal{B} = -\frac{1}{3}$. All other particles are assumed to have zero baryon number and hence do not transform under this symmetry operation. One can go back and check easily that no interaction of the standard model changes a quark into a lepton, hence this symmetry.

In the leptonic sector also there are symmetries of this kind. For example, consider

$$\ell \to e^{-i\mathcal{L}\theta} \ell\,, \tag{15.114}$$

where ℓ is any negatively charged lepton or its corresponding neutrino. Conventionally, the quantity $\mathcal{L}$, called *lepton number*, is taken to be 1 for all of them and -1 for their antiparticles. All other particles have zero lepton number. In fact, unlike the quarks, there is no intergenerational mixing in the leptonic sector of the standard model, so even the lepton numbers corresponding to each generation

is conserved. In other words, if we assign a global quantum number 1 to the electron and the electron-neutrino, -1 for their antiparticles, and zero to all other particles, the quantum number will be conserved. The same applies if we assign the number to any other charged lepton and its associated neutrino only.

These symmetries have a very different character than the gauge symmetry. First of all, these are global symmetries, as opposed to the gauge symmetry which is local. Secondly, these are not symmetries that we imposed on the model. We built up the theory with the gauge symmetry and with the specific fermion content, and after doing that we found these extra symmetries in the theory. Such symmetries are called *accidental symmetries*.

There is another difference. In this book, we have always relied on perturbation theory to derive the interactions. However, if non-perturbative effects are allowed, it can be shown that a global U(1) quantum number $\mathscr{G}$ cannot be a symmetry of the model unless it satisfies the relations

$$\left(\sum_L - \sum_R\right) \mathscr{G}^2 \mathscr{Q} = 0\,, \qquad \left(\sum_L - \sum_R\right) \mathscr{G} \mathscr{Q}^2 = 0\,. \quad (15.115)$$

The two sums in this equation are over the left-chiral and the right-chiral fermions, and $\mathscr{Q}$ is any diagonal local charge. The general condition is somewhat more involved, but this is enough to show that baryon number and lepton number are not symmetries if we include non-perturbative effects. However, the difference $\mathcal{B} - \mathcal{L}$ is still a global symmetry in the model, since it satisfies all the tests like the ones given in Eq. (15.115).

□ **Exercise 15.20** Verify that

$$\left(\sum_L - \sum_R\right) (\mathcal{B} - \mathcal{L})^2 \mathscr{Q} = 0\,,$$
$$\left(\sum_L - \sum_R\right) (\mathcal{B} - \mathcal{L}) \mathscr{Q}^2 = 0\,, \quad (15.116)$$

where $\mathscr{Q}$ can be the electric charge or the hypercharge Y. [Hint: Remember that each quark is actually a color triplet.]

Appendix A

Useful formulas

A.1 Representation of γ-matrices

The γ-matrices are defined by the anticommutation relation

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}\,, \tag{A.1}$$

and their hermiticity properties:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,. \tag{A.2}$$

We mentioned that these two properties do not uniquely define the specific form of the matrices. In particular, if two sets of matrices $\{\gamma^\mu\}$ and $\{\tilde{\gamma}^\mu\}$ both satisfy Eqs. (A.1) and (A.2), they will be related by

$$\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger\,, \tag{A.3}$$

for some unitary matrix U. In view of this fact, we tried to do everything in a way that is manifestly independent of the representation of the γ-matrices.

In order to gain some insight into the solutions, sometimes a specific representation is helpful. For example, in §4.3.2, we used the Dirac-Pauli representation, in which the matrices are given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,, \tag{A.4}$$

where I is the 2×2 unit matrix and the σ^i's are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{A.5}$$

The advantage of this representation is understood by looking at the solutions for the spinors derived in §4.3.2. For a non-relativistic particle, the lower two components of the u-spinors are negligible. Hence the non-relativistic spinor is effectively a two-component one, which is easier to use. Thus the Dirac-Pauli representation is particularly helpful for the non-relativistic reduction of a problem. It was used for this purpose in §11.2. In this representation, γ_5 and $\sigma_{\mu\nu}$ take the forms

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \sigma^{0i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \sigma_{ij} = \varepsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{A.6}$$

where ε_{ijk} is completely antisymmetric symbol with $\varepsilon_{123} = +1$.

Another useful representation is called the *chiral representation*, in which

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{A.7}$$

Notice that the matrices γ^i have the same form as in the Dirac-Pauli representation, but γ^0 is different. In this representation,

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{A.8}$$

Thus the chirality projection operators in this representation have the form

$$\frac{1}{2}(1+\gamma_5) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{A.9}$$

So the chiral eigenstates will have zeros for either the upper two components, or the lower two. If one has to deal with chiral eigenstates explicitly, this is a more convenient representation to use. It is used extensively, for example, in discussing supersymmetric field theories, a subject which is beyond the scope of the present book. In this representation, σ_{ij} are the same as in the Dirac-Pauli representation, whereas

$$\sigma^{0i} = i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}. \tag{A.10}$$

A third useful representation is the *Majorana representation.* Here,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad , \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \quad , \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} , \qquad \text{(A.11)}$$

from which we can calculate

$$\gamma_5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} . \qquad \text{(A.12)}$$

Notice that all the γ-matrices are purely imaginary. For one thing, it assures that the $\sigma_{\mu\nu}$'s are also purely imaginary. In the Dirac equation, the operator $i\gamma^\mu\partial_\mu - m$ is now completely real. As a result, if we have a solution where all the components are real at one time, they will remain real for all subsequent times. This is very useful if there are fermionic particles which are antiparticles of themselves.

□ **Exercise A.1** Find $\sigma^{\mu\nu}$'s in the Majorana representation.

A.2 Traces of γ-matrices

For the sake of notational simplicity, we will use a number of 4-vectors which we will denote by a_r, with $r = 1, 2, 3\cdots$, and consider traces of the form

$$T_n(a_1, a_2, \cdots a_n) \equiv \text{Tr}\,[\not{a}_1\not{a}_2\cdots\not{a}_n] . \qquad \text{(A.13)}$$

We first show that T_n vanishes if n is odd. For this, we use the properties of the matrix γ_5. Since its square is the unit matrix, we can write

$$\begin{aligned} T_n(a_1, a_2, \cdots a_n) &= \text{Tr}\,[\gamma_5\gamma_5\not{a}_1\not{a}_2\cdots\not{a}_n] \\ &= \text{Tr}\,[\gamma_5\not{a}_1\not{a}_2\cdots\not{a}_n\gamma_5] , \end{aligned} \qquad \text{(A.14)}$$

using the cyclic property of traces. We now take the γ_5 appearing at the end through the $\not{a}_k$'s. Since γ_5 anticommutes with all the γ_μ's, we will obtain a minus sign each time we go through one of the slashed vectors. To go through n of them, we will obtain a factor

$(-1)^n$, at which point the two factors of γ_5 will be sitting next to each other, and we will forget them since those two factors will give the unit matrix. Thus we obtain

$$T_n = (-1)^n T_n \,. \tag{A.15}$$

This equation says nothing if n is even, but if n is odd, it proves that T_n vanishes.

The non-trivial traces thus involve an even number of γ-matrices. For the case of $n = 2$, we obtained the trace in §7.2.1 at some length. Here we repeat the derivation in a different notation, which will be useful for the traces with higher number of γ-matrices. For any two vectors a and b,

$$\not{a}\not{b} = 2a \cdot b - \not{b}\not{a} \,, \tag{A.16}$$

which follows from the anticommutation relation of the γ-matrices. Therefore,

$$\begin{aligned} T_2(a_1, a_2) &= \mathrm{Tr}\,[\not{a}_1\not{a}_2] \\ &= \mathrm{Tr}\,[2a_1 \cdot a_2] - \mathrm{Tr}\,[\not{a}_2\not{a}_1] \\ &= \mathrm{Tr}\,[2a_1 \cdot a_2] - \mathrm{Tr}\,[\not{a}_1\not{a}_2] \,, \end{aligned} \tag{A.17}$$

where we have used the cyclic property of traces. By rearranging the terms, we obtain

$$T_2(a_1, a_2) = \mathrm{Tr}\,[a_1 \cdot a_2] = a_1 \cdot a_2 \mathrm{Tr}\,[\mathbf{1}] \,, \tag{A.18}$$

where we have explicitly written the unit matrix implied in Eq. (A.16). The trace of the unit matrix is 4, so finally we get

$$T_2 = 4a_1 \cdot a_2 \,. \tag{A.19}$$

Once this is done, all higher traces can be expressed in terms of T_2. For example, starting now with the definition of T_4 and using this relation repeatedly, we obtain

$$\begin{aligned} T_4(a_1, a_2, a_3, a_4) &\equiv \mathrm{Tr}\,[\not{a}_1\not{a}_2\not{a}_3\not{a}_4] \\ &= \mathrm{Tr}\,[(2a_1 \cdot a_2 - \not{a}_2\not{a}_1)\not{a}_3\not{a}_4] \\ &= 2a_1 \cdot a_2\, T_2(a_3, a_4) - \mathrm{Tr}\,[\not{a}_2\not{a}_1\not{a}_3\not{a}_4] \\ &= 2a_1 \cdot a_2\, T_2(a_3, a_4) - \mathrm{Tr}\,[\not{a}_2(2a_1 \cdot a_3 - \not{a}_3\not{a}_1)\not{a}_4] \\ &= 2a_1 \cdot a_2\, T_2(a_3, a_4) - 2a_1 \cdot a_3\, T_2(a_2, a_4) + \mathrm{Tr}\,[\not{a}_2\not{a}_3\not{a}_1\not{a}_4] \\ &= 2a_1 \cdot a_2\, T_2(a_3, a_4) - 2a_1 \cdot a_3\, T_2(a_2, a_4) \\ &\quad + 2a_1 \cdot a_4\, T_2(a_2, a_3) - \mathrm{Tr}\,[\not{a}_2\not{a}_3\not{a}_4\not{a}_1] \,. \end{aligned} \tag{A.20}$$

The remaining trace is equal to the original trace due to the cyclic property of traces. So we obtain

$$\begin{aligned} & T_4(a_1, a_2, a_3, a_4) \\ & \quad = a_1 \cdot a_2 \, T_2(a_3, a_4) - a_1 \cdot a_3 \, T_2(a_2, a_4) + a_1 \cdot a_4 \, T_2(a_2, a_3) \\ & \quad = 4 \, (a_1 \cdot a_2 \, a_3 \cdot a_4 - a_1 \cdot a_3 \, a_2 \cdot a_4 + a_1 \cdot a_4 a_2 \cdot a_3) \, . \end{aligned} \tag{A.21}$$

In a similar way, we can show that

$$\begin{aligned} T_{2n}(a_1, a_2, \cdots a_{2n}) = \; & a_1 \cdot a_2 \, T_{2n-2}(a_3, a_4, \cdots a_{2n}) \\ & - a_1 \cdot a_3 \, T_{2n-2}(a_2, a_4, \cdots a_{2n}) \\ & + a_1 \cdot a_4 \, T_{2n-2}(a_2, a_3, a_5 \cdots a_{2n}) + \cdots \, . \end{aligned} \tag{A.22}$$

The traces can also be expressed in their bare form, without using the 4-vectors. For example, from Eq. (A.19), we can extract the coefficients of $a_1^\mu a_2^\nu$ from both sides to write

$$\mathrm{Tr}\,[\gamma_\mu \gamma_\nu] = 4 g_{\mu\nu} \, . \tag{A.23}$$

Similarly, from Eq. (A.21), we can take the co-efficients of $a_1^\mu a_2^\nu a_3^\lambda a_4^\rho$ to obtain

$$\mathrm{Tr}\,[\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho] = 4 \, (g_{\mu\nu} g_{\lambda\rho} - g_{\mu\lambda} g_{\nu\rho} + g_{\mu\rho} g_{\nu\lambda}) \, . \tag{A.24}$$

Since γ_5 contains four γ-matrices, traces involving γ_5 can be non-vanishing only if there is an even number of γ-matrices multiplying γ_5. If this number happens to be zero, the trace is zero, since we have shown in Ch. 4 that γ_5 is traceless. The trace also vanishes if γ_5 is accompanied by two γ-matrices. This is because

$$\begin{aligned} \mathrm{Tr}\,[\gamma_\mu \gamma_\nu \gamma_5] & = - \mathrm{Tr}\,[\gamma_\mu \gamma_5 \gamma_\nu] \\ & = - \mathrm{Tr}\,[\gamma_\nu \gamma_\mu \gamma_5] \, . \end{aligned} \tag{A.25}$$

At the first equality sign, we have used the anticommuting property of γ_5, and at the second, we have used the cyclic property of traces. What we have obtained is that the trace should be antisymmetric in the indices μ, ν. There is no such rank-2 tensor in the structure of space-time, and hence the trace must vanish. The smallest non-vanishing trace involving γ_5 thus contains four more γ-matrices.

These four have to be all different, because if any two are the same, the trace can be reduced to the form in Eq. (A.25). Because difference γ-matrices anticommute, the trace has to be proportional to the antisymmetric tensor. The proportionality constant can be fixed by calculating $\mathrm{Tr}\,[\gamma_0\gamma_1\gamma_2\gamma_3\gamma_5]$.

$$\mathrm{Tr}\,[\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_5] = 4i\epsilon_{\mu\nu\lambda\rho}\,. \tag{A.26}$$

To obtain higher traces involving γ_5, it is useful to use the identity

$$\gamma_\mu\gamma_\nu\gamma_\lambda = g_{\mu\nu}\gamma_\lambda - g_{\mu\lambda}\gamma_\nu + g_{\nu\lambda}g_\mu - i\epsilon_{\mu\nu\lambda\rho}\gamma^\rho\gamma_5\,, \tag{A.27}$$

which can be checked by substitution.

One interesting property of the γ-matrix traces can be proved using the matrix C introduced in §10.3. This is

$$\mathrm{Tr}\,[\not{a}_1\not{a}_2\cdots\not{a}_{2n-1}\not{a}_{2n}] = \mathrm{Tr}\,[\not{a}_{2n}\not{a}_{2n-1}\cdots\not{a}_2\not{a}_1]\,. \tag{A.28}$$

□ **Exercise A.2** Prove Eq. (A.28).

□ **Exercise A.3** Show that any matrix which anticommutes with all the γ_μ must be a multiple of γ_5. [Hint: Use the basis matrices given in Eq. (4.14).]

A.3 The antisymmetric tensor

The antisymmetric, or the Levi-Civita tensor, is a rank-4 tensor which is antisymmetric in the interchange of any pair of indices. Thus, the tensor can have non-zero components only when all the four indices are different, and in this case, if we define its value for any order of the indices, the antisymmetry fixes the values for all others. We have chosen

$$\epsilon_{0123} = +1\,. \tag{A.29}$$

Following the usual rules for raising and lowering of indices, this would also imply

$$\epsilon^{0123} = -1\,. \tag{A.30}$$

When there is a pair of antisymmetric tensors, we can express them in terms of the metric tensor:

$$\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu'\nu'\lambda'\rho'} = -g^{\mu}_{[\mu'}g^{\nu}_{\nu'}g^{\lambda}_{\lambda'}g^{\rho}_{\rho']}\,, \tag{A.31}$$

where the square brackets indicate that all permutations of the lower indices are to be added, even permutations with a factor of $+1$ and odd permutations with a factor of -1. This can be verified by direct substitution. When one index is contracted, we obtain

$$\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu'\lambda'\rho'} = -\Big(g^{\nu}_{\nu'}g^{\lambda}_{\lambda'}g^{\rho}_{\rho'} + g^{\nu}_{\lambda'}g^{\lambda}_{\rho'}g^{\rho}_{\nu'} + g^{\nu}_{\rho'}g^{\lambda}_{\nu'}g^{\rho}_{\lambda'} - g^{\nu}_{\lambda'}g^{\lambda}_{\nu'}g^{\rho}_{\rho'} - g^{\nu}_{\rho'}g^{\lambda}_{\lambda'}g^{\rho}_{\nu'} - g^{\nu}_{\nu'}g^{\lambda}_{\rho'}g^{\rho}_{\lambda'} \Big) . \quad \text{(A.32)}$$

We can now go further and contract one more index. This will give

$$\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda'\rho'} = -2\left(g^{\lambda}_{\lambda'}g^{\rho}_{\rho'} - g^{\lambda}_{\rho'}g^{\rho}_{\lambda'} \right) . \quad \text{(A.33)}$$

Another contraction would lead to the identity

$$\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda\rho'} = -6g^{\rho}_{\rho'} , \quad \text{(A.34)}$$

and finally, if all the indices are contracted, we obtain

$$\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda\rho} = -24 . \quad \text{(A.35)}$$

A.4 Useful integration formulas

A.4.1 Angular integrations in N-dimensional space

Consider an N-dimensional space with Cartesian co-ordinates $x_1, x_2 \cdots x_N$. Let us denote the magnitude of $\boldsymbol{x}$ by r, i.e.,

$$\sum_{i=1}^{N} x_i^2 = r^2 . \quad \text{(A.36)}$$

If we integrate a function of r over the entire space, we should be able to express the integral in the form

$$\int d^N x \; f(r) = C_N \int_0^\infty dr \; r^{N-1} f(r) \quad \text{(A.37)}$$

for some number C_N. Essentially, this C_N is obtained by integrating over the angular variables in a spherical co-ordinate system. Our task is to determine this C_N.

Since the angular integrations do not depend on the form of the function $f(r)$, we can use any convenient function to obtain C_N. Let us consider

$$f(r) = \exp\left(-r^2/a^2\right) . \tag{A.38}$$

In this case, the integral appearing on the right side of Eq. (A.37) can be rewritten as

$$\int_0^\infty dr\, r^{N-1} \exp\left(-r^2/a^2\right) = a^N \int_0^\infty d\rho\, \rho^{N-1} \exp\left(-\rho^2\right) . \tag{A.39}$$

In mathematical analysis, the Γ-function for any complex variable z is defined by the relation

$$\begin{aligned} \Gamma(z) &\equiv \int_0^\infty d\zeta\, \zeta^{z-1} e^{-\zeta} \\ &= 2\int_0^\infty d\rho\, \rho^{2z-1} e^{-\rho^2} , \end{aligned} \tag{A.40}$$

where we have written $\zeta = \rho^2$ in the last step. Thus we obtain the result:

$$\text{Right hand side of Eq. (A.37)} = \frac{1}{2} C_N a^N \Gamma(N/2) . \tag{A.41}$$

On the other hand, using Eq. (A.36), we can write the left side in the form

$$\begin{aligned} \int d^N x\, \exp\left(-\sum_i x_i^2/a^2\right) &= \left[\int_{-\infty}^\infty dx\, \exp\left(-x^2/a^2\right)\right]^N \\ &= a^N \left[\Gamma(1/2)\right]^N . \end{aligned} \tag{A.42}$$

For $N = 2$, Eqs. (A.41) and (A.42) give $\Gamma(1/2) = \sqrt{\pi}$. For arbitrary N, we then obtain

$$C_N = \frac{2\pi^{N/2}}{\Gamma(N/2)} . \tag{A.43}$$

□ **Exercise A.4** If we considered

$$f(r) = \begin{cases} 1 & \text{if } r \le R , \\ 0 & \text{if } r > R , \end{cases} \tag{A.44}$$

the integration defined in Eq. (A.37) should have given us the volume of a sphere in N-dimension. Use the formula for C_N given above to verify that one obtains the correct "volume" for $N = 2, 3$.

A.4.2 Momentum integration in loops

In loop diagrams, after introducing the Feynman parameters, the momentum integrations can be reduced to the form

$$I_s^r(a,N) = \int \frac{d^N k}{(2\pi)^N} \frac{(k^2)^r}{(k^2-a^2)^s} . \tag{A.45}$$

Here, N is the dimension of space-time. For convergent integrals, it can be taken as 4. In order to tackle divergent integrals as well, a subject which has been discussed in Ch. 12, we keep N arbitrary here.

After performing the Wick rotation, the integral becomes

$$I_s^r(a,N) = i(-1)^{r+s} \int \frac{d^N k_E}{(2\pi)^N} \frac{(k_E^2)^r}{(k_E^2+a^2)^s} . \tag{A.46}$$

At this point, we can use the result obtained above for the integration over the angular variables to write

$$I_s^r(a,N) = \frac{i(-1)^{r+s}}{(4\pi)^{N/2}\Gamma(N/2)} \frac{1}{(a^2)^{s-r-\frac{1}{2}N}} J_s^r(N) , \tag{A.47}$$

where

$$J_s^r(N) = \int_0^\infty dy \, \frac{y^{r-1+\frac{1}{2}N}}{(y+1)^s} , \tag{A.48}$$

with $y = k_E^2/a^2$. This remaining integral can be expressed in a standard form if we substitute

$$z = \frac{1}{y+1} . \tag{A.49}$$

In that case,

$$1-z = \frac{y}{y+1} , \qquad dz = -\frac{dy}{(y+1)^2} . \tag{A.50}$$

So we can write

$$\begin{aligned} J_s^r(N) &= \int_0^\infty \frac{dy}{(y+1)^2} \left(\frac{y}{y+1}\right)^{r-1+\frac{1}{2}N} \left(\frac{1}{y+1}\right)^{s-r-1-\frac{1}{2}N} \\ &= \int_0^1 dz \, (1-z)^{r-1+\frac{1}{2}N} z^{s-r-1-\frac{1}{2}N} \\ &= B\left(r+\frac{1}{2}N, s-r-\frac{1}{2}N\right) . \end{aligned} \tag{A.51}$$

In the last step, we have introduced the beta-function of mathematical analysis, which can be given in terms of the Γ-function defined above:

$$B(z,z') = \frac{\Gamma(z)\Gamma(z')}{\Gamma(z+z')}\,. \tag{A.52}$$

Substituting this into Eq. (A.47), we finally obtain

$$I^r_s(a,N) = \frac{i(-1)^{r+s}}{(4\pi)^{N/2}}\,\frac{1}{(a^2)^{s-r-\frac{1}{2}N}}\,\frac{\Gamma(r+\frac{1}{2}N)\Gamma(s-r-\frac{1}{2}N)}{\Gamma(\frac{1}{2}N)\Gamma(s)}\,. \tag{A.53}$$

For convergent integrals, we can put $N=4$ as mentioned earlier and suppress the dimension. This gives

$$I^r_s(a) = \frac{i(-1)^{r+s}}{(4\pi)^2}\,\frac{1}{(a^2)^{s-r-2}}\,\frac{\Gamma(r+2)\Gamma(s-r-2)}{\Gamma(s)}\,. \tag{A.54}$$

Appendix B

Answers to selected exercises

Ex. 1.3 •

$$\text{Hamiltonian :} \qquad H = \sum_{i=1}^{N} \left(a_i^\dagger a_i + \frac{1}{2} \right) \hbar \omega_i \,,$$

$$\text{state :} \qquad |n_1, n_2, \cdots, n_N\rangle = \prod_{i=1}^{N} \frac{\left(a_i^\dagger\right)^{n_i}}{\sqrt{n_i!}} |0\rangle \,.$$

$$\text{number operator :} \qquad \mathcal{N} = \sum_{i=1}^{N} a_i^\dagger a_i \,.$$

Ex. 1.7 • Maxwell equations are given in Ch. 8. The Lorentz force law on a particle with charge q is

$$\frac{dp^\mu}{d\tau} = q F^{\mu\nu} \frac{dx_\nu}{d\tau} \,,$$

where x^μ are the coordinates of the worldline of the particle, and τ is proper time, $d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$.

Ex. 1.8 • $\tau = 2 \times 10^{-6}\, s$.

Ex. 2.2 • $(\Box + m^2)\phi = -\dfrac{\partial V}{\partial \phi}$.

Ex. 2.3 • $(\Box + m^2)\phi^\dagger = -\dfrac{\partial V}{\partial \phi}$.

Ex. 2.4 • The answer appears in Ch. 8.

Ex. 2.5 • $\Pi^i = \dot{A}^i$.

Ex. 2.6 • $\mathscr{H} = |\dot{\phi}|^2 + |\boldsymbol{\nabla}\phi|^2 + m^2 \phi^\dagger \phi + V(\phi^\dagger \phi)$.

Ex. 2.7 •

$$a)\quad p_k = l\dot{q}_k\,.$$

$$b)\quad H = \frac{1}{2l}\sum_{k=1}^{\infty}\left(p_k^2 + l^2\omega_k^2 q_k^2\right).$$

$$c)\quad [a_k, a_m^\dagger]_- = \delta_{km}\,,\quad [a_k, a_m]_- = [a_k^\dagger, a_m^\dagger]_- = 0\,.$$

$$d)\quad H = \sum_{k=1}^{\infty}\frac{\hbar\omega_k}{2}(a_k a_k^\dagger + a_k^\dagger a_k)\,.$$

Ex. 2.8 •

$$a)\quad T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathscr{L}\,,$$

$$b)\quad T^{\mu\nu} = \partial^\mu\phi^\dagger\partial^\nu\phi + \partial^\mu\phi\partial^\nu\phi^\dagger - g^{\mu\nu}\mathscr{L}\,,$$

$$c)\quad T^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho - g^{\mu\nu}\mathscr{L}\,.$$

Ex. 2.9 •

$$a)\quad j^\mu = iq(\phi^\dagger\partial^\mu\phi - \phi\partial^\mu\phi^\dagger)\,.$$

$$b)\quad j^\mu = iq(\phi^\dagger\partial^\mu\phi - \phi\partial^\mu\phi^\dagger + 2iqA^\mu\phi^\dagger\phi)\,.$$

Ex. 2.10 • $j^\mu = \partial^\mu(x_\nu\phi)\partial^\nu\phi - x^\mu\mathscr{L}$.

Ex. 3.5 • $P^\mu = \frac{1}{2}\int d^3p\, p^\mu\left(a^\dagger(p)a(p) + a(p)a^\dagger(p)\right)$.

Ex. 3.7 • $Q = iq\int d^3p\left[a_1^\dagger(p)a_2(p) - a_2^\dagger(p)a_1(p)\right]$.

Ex. 4.8 • $m\overline{v}_r(\boldsymbol{p})\gamma^\mu v_s(\boldsymbol{p}) = -p^\mu\overline{v}_r(\boldsymbol{p})v_s(\boldsymbol{p})$.

Ex. 4.9 • $\overline{v}(\boldsymbol{p}')\,\gamma^\mu\,v(\boldsymbol{p}) = -\frac{1}{2m}\,\overline{v}(\boldsymbol{p}')\,[(p+p')^\mu - i\sigma^{\mu\nu}q_\nu]\,v(\boldsymbol{p})$.

Ex. 4.12 • Same as in Eq. (4.61) and Eq. (4.62).

Ex. 4.26 • $\mathscr{M}^\lambda{}_{\mu\nu} = \frac{1}{2}\overline{\psi}\gamma^\lambda\sigma_{\mu\nu}\psi - \left(T^\lambda{}_\mu x_\nu - T^\lambda{}_\nu x_\mu\right)$, where $T^\lambda{}_\mu = \overline{\psi}i\gamma^\lambda\partial_\mu\psi - g^\lambda{}_\mu\mathscr{L}$.

Ex. 6.2 • $\dfrac{G_\beta}{\sqrt{2}}\,\overline{\psi}_{(\nu_r)}\gamma^\mu(1-\gamma_5)\psi_{(e)}\,\overline{\psi}_{(n)}\gamma_\mu(1-\gamma_5)\psi_{(p)}$.

Ex. 6.4 •

$$S_{fi}^{(3c)} = (-ih)^3\,(2\pi)^4\delta^4(k-p-p')\left[\frac{1}{\sqrt{2\omega_k V}}\,\frac{1}{\sqrt{2E_p V}}\,\frac{1}{\sqrt{2E_{p'}V}}\right]$$
$$\times\int\frac{d^4q}{(2\pi)^4}\,i\Delta_F(q)\,[\overline{u}_s(\boldsymbol{p})iS_F(p-q)\,iS_F(p)v_{s'}(\boldsymbol{p}')]\,,$$

$$S_{fi}^{(3d)} = (-ih)^3\,(2\pi)^4\delta^4(k-p-p')\left[\frac{1}{\sqrt{2\omega_k V}}\,\frac{1}{\sqrt{2E_p V}}\,\frac{1}{\sqrt{2E_{p'}V}}\right] \times \int \frac{d^4q}{(2\pi)^4}\; i\Delta_F(q)\,[\bar{u}_s(\boldsymbol{p})iS_F(p)\;iS_F(p-q)v_{s'}(\boldsymbol{p}')]\,.$$

Ex. 7.3 • Yes.

Ex. 7.5 • $\Gamma = \dfrac{G_F^2 m_\mu^5}{192\pi^3}\left(1 - 8\dfrac{m_e^2}{m_\mu^2} + \cdots\right)$.

Ex. 7.6 • $\dfrac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \dfrac{m_e^2}{m_\mu^2}\left(\dfrac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2$.

Ex. 7.9 • 11 GeV approximately.

Ex. 8.1 • $\partial_\mu F^{\mu\nu} + M^2 A^\nu = j^\nu$.

Ex. 8.2 • $\theta(x) = \theta_0(x) + \int d^4x'\; G_0(x-x')f(x')$, where G_0 is the massless scalar propagator (Green's function for the wave equation) and θ_0 satisfies the wave equation.

Ex. 8.5 • Physical states: $a_r^\dagger(k)\,|0\rangle$ for $r = 1, 2$.

Ex. 8.6 • $:P^\mu: \;= \int d^3k\; k^\mu \sum_r a_r^\dagger(k)a_r(k)$, where $k^0 = \omega_k$. This expression is valid on physical states selected by the Gupta-Bleuler condition, so that only the transverse modes contribute for each k^μ in the integral. (Physical quantities like Hamiltonian and momentum are independent of ξ when restricted to physical states.)

Ex. 11.3 • $eF(0)$.

Ex. 12.3 • Define $E_{\mu\alpha|\nu\beta} = g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\nu\alpha}$. Then $\pi_{\mu\nu\lambda\rho}(k_1,k_2,k_3,k_4) = ak_1^\alpha k_2^\beta k_3^\gamma k_4^\delta\Big(E_{\mu\alpha|\nu\beta}E_{\lambda\gamma|\rho\delta} + E_{\mu\alpha|\lambda\gamma}E_{\nu\beta|\rho\delta} + E_{\mu\alpha|\rho\delta}E_{\nu\beta|\lambda\gamma}\Big) + bk_1^\alpha k_2^\beta k_3^\gamma k_4^\delta\Big(E_{\mu\alpha|\mu'\nu'}E^{\nu'\lambda'}_{\nu\beta}E_{\lambda\gamma|\lambda'\rho'}E^{\rho'\mu'}_{\rho\delta} + E_{\mu\alpha|\mu'\nu'}E^{\nu'\lambda'}_{\lambda\gamma}E_{\nu\beta|\lambda'\rho'}E^{\rho'\mu'}_{\rho\delta} + E_{\mu\alpha|\mu'\nu'}E^{\nu'\lambda'}_{\nu\beta}E_{\rho\delta|\lambda'\rho'}E^{\rho'\mu'}_{\lambda\gamma}\Big)$, where a and b are two form factors which depend on all possible Lorentz invariant combinations of the momenta.

Ex. 12.8 • 1/128 approximately.

Ex. 13.12 • $Q_a = \varepsilon_{abc}\int d^3x\;\dot{\phi}_b\phi_c$.

Ex. 15.13 • −17%.

Ex. 15.19 • +1.9%

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AMITABHA LAHIRI
S.N. Bose National Centre for Basic Sciences
Salt Lake, Calcutta 700 098, India

PALASH B. PAL
Saha Institute of Nuclear Physics
Salt Lake, Calcutta 700 064, India

This book introduces QFT for a reader with no prior knowledge of the subject. It is meant to be a textbook for advanced undergraduate or beginning postgraduate students. The book discusses quantization of fields, S-matrix theory, Feynman diagrams, calculation of decay rates and cross sections, renormalization, symmetries and symmetry breaking. Some background material on classical field theory and group theory, needed for the exposition, are also presented in the book. Detailed calculations of weak and electromagnetic processes are included. There are many exercise problems to help the students, instructors and beginning researchers in the field.

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